Space-time reasoning in logic

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Every phenomenon of the physical world occurs in space and time; these are the two fundamental forms of any motion. Though subjectivity does not obey any spatial or temporal limitations in any straightforward way, it has to be somehow implemented in a system of interacting material bodies, and every such implementation is bound to reflect the space-time properties of the material used. In particular, logical reasoning in humans must manifest a combination of both spatial and temporal aspects in every single act, and one of them may dominate sometimes over the other.

Since any object can be a part of many higher-level systems, it can manifest quite different features when viewed in respect to a specific external process. The opposition of space and time is only one of the possible schemes, and its representation in human reasoning can in no way exhaust the possible manifestations. Though some types of complementary descriptions (like geometric and dynamic methods in analytical mechanics, functional and operator forms of quantum mechanics *etc.*) could be related to the opposition of space and time, there are other kinds of complementarity apparently unrelated to this issue, such as, for example, the juxtaposition of configuration space and phase space pictures of dynamics. In this latter case, one will find a kind of space-time duality: motion in the configuration space forms the geometry of phase space, so that a dynamic (time-dependent) picture gets complemented with a static (space-like) description. A point in the phase space may correspond to a straight trajectory in the configuration space; conversely, two points in the phase space (fixing a straight line) determine a point in the conjugate configuration space (the intersection of two trajectories).

One could also recall the two paradigms of statistical physics, where one finds both time averages and averaging over statistical ensembles, and it is only in an ergodic system that the two become equal.

In logic, too, there are complementary aspects of any particular act of reasoning, one resembling space, and the other seemingly akin to time. Considering the universe of established facts (for instance, a universe of true sentences of propositional logic, or a collection of axioms and theorems of a mathematical theory), one could extend it in two opposite ways, either adding a new arbitrary assertion or deriving a consequence of the already existing facts as a hypothesis yet to be recognized as a new fact. The former (extensive) way is readily associated with spatial expansion, while the latter (progressive) way manifests a serial organization similar to that of physical time. The two directions of logical development are relatively independent, since adding new “true” sentences (axioms) does not require any inference, while the application of the pre-defined inference rules does not require any new axioms to proceed. This resembles the “orthogonality” of space and time coordinates in classical physics. However, many physical processes obey the principle of contingence, which could be compared with logical consistency: in a system with stationary dynamics, the possible trajectories cover the whole configuration space, so that every point in the system’s configuration space can only be achieved from the points lying on a certain trajectory. This holds for either mechanical (both classical and quantum) or non-mechanical (thermodynamic and other) systems, with the appropriate redefinition of space and time variables. In the same way, formal logical reasoning is bound to follow certain lines of reasoning determined by the system of the adopted inference rules. Two sentences are logically independent (within a given inference system) if they do not lie on the same logical trajectory, that is, they cannot be inferred from each other. This means that the addition of a new axiom or definition requires a consistency check against the already accepted statements: new space points have to be either dynamically achieved or unachievable in the current dynamic conditions.

The problem with this analogy is that logical “trajectories” do not quite resemble the trajectories of classical mechanics, since every conclusion is drawn from at least two statements rather than from a single one. However, one can indicate that the difference is entirely superficial, and the analogy between logical reasoning and physical processes is much closer, being deep-rooted and essential.

The traditional logic of most scientific theories (including mathematics) is largely based on the same fundamental figure of syllogism known as *modus ponendo ponens*, or simply *modus ponens*. That is, demonstrating, for instance, that *modus ponens* is constructed in a space-time manner of classical physics, we picture any scientific discourse as an analog of physical motion, sharing its space-time structure. As it is well known, *modus ponens* binds together three sentences (propositions) of a special structure:

The minor premise: *S* **is** *M*
The major premise: **All** *M* **are** *P*
The conclusion: *S* **is** *P*

Under certain conditions, it could be shown that all the other figures of syllogism can be unfolded from *modus ponens* (which may require additional premises); most frequently, we assume a kind of completeness, or the existence of an exhausting class.

In the above construction, the major premise is of a different structure than the minor premise and the conclusion. This suggests analogy between a universal statement (containing the universal quantifier "all", whatever it means) and an operator, or a transformation rule, which would not belong to the collection of sentences it is applied to. In this respect, *modus ponens* appears to connect one sentence (the minor premise) to another (the conclusion) by means of an operator (the major premise). This exactly corresponds to the scheme of analytical mechanics, where a point of a manifold gets connected to an adjacent point through an element of the tangent space attached to the manifold at the original point, which is known to be a differential operator corresponding to the vector of velocity. To put it plain, one obtains one spatial point from another using a difference operator:

$$x' = x + \Delta x,$$

which becomes differential in the infinitesimal case:

$$x' = x + dx.$$

A combination of a spatial position and momentum (roughly proportional to velocity, or, in general, being some linear combinations of the elements of the tangent space) represents a point of the system's phase space. In this way, the three levels of modus ponens reproduce the structure of classical mechanics, with specific assertion treated as the points of a configuration space and general statements forming the corresponding phase space. In mechanics, the same point can be linked to a range of other points by different operators from the tangent space; similarly, different people draw different conclusions from the same premises in different situations. For instance, one might consider the logical analogs of longitudinal and transverse displacements. Thus, in a two-dimensional space, one can distinguish the acts like of

$$(x_1, y) = (x, y) + (\Delta x_1, 0)$$
$$(x_2, y) = (x_1, y) + (\Delta x_2, 0)$$

from the combined shifts in orthogonal dimensions:

$$(x', y) = (x, y) + (\Delta x, 0)$$
$$(x, y') = (x, y) + (0, \Delta y)$$

The former case corresponds to the chain of logical inference

$$S \textbf{ is } M \to (\textbf{All } M \textbf{ are } M') \to S \textbf{ is } M' \to (\textbf{All } M' \textbf{ are } P) \to S \textbf{ is } P$$

In this manner, one can construct "proofs" of any length, always remaining within the serial (deductive) paradigm. To introduce yet another dimension, we have to consider "logical parallels" treating the different aspects of the same in a similar way:

$$S \textbf{ is } M_1 \to (\textbf{All } M_1 \textbf{ are } P_1) \to S \textbf{ is } P_1$$
$$S \textbf{ is } M_2 \to (\textbf{All } M_2 \textbf{ are } P_2) \to S \textbf{ is } P_2$$

Depending on the nature of thus distinguished attributes, the geometry of resulting many-dimensional may be rather complicated, just like in mechanical systems. The distinction is obviously relative, as the same conclusion be achieve in many ways. Here, once again, we encounter the interplay of temporal (serial) and spatial (extensive) paradigms that can be combined in any logical discourse.

Since the application of any inference rule is local, the manifold of logical reasoning (the inference space) may have rather complex topology and geometry, comparable to those often found in analytical mechanics. Further, extending the analogy to quantum mechanics, one could consider all kinds of "alternative" modes of reasoning.

In this picture, an illegal extrapolation of local properties to the whole inference space may lead to logical fallacies and incorrectly reproduce the global structure of inference space in formal theories. Thus, the traditional introduction of logical *negation* is based on the assumption of a nearly Euclidean global geometry, without torsion, cusps, lacunas and other singularities. In such a "smooth" space individual trajectories do not intersect, they do not form loops, and the order of trajectories in any direction remains the same along any trajectory. This may not hold for many practical cases studied, say, by social sciences, and especially for non-scientific modes of reasoning.

The problem of *completeness* is much akin to the problem of meaningful negation. For example, the presence of a pole singularity in the inference space of a theory will result in that no finite trajectory can ever reach the point of singularity, though this point may be quite reachable in an "embedding" inference space with a simpler topology. Singularities in the phase space may hinder reaching an obviously valid conclusion via a finite inference; a different formulation of the same theory will make deduction possible, while cutting the logical routes to some earlier established facts. Once again, to restore logical completeness, we need to consider a wider inference space, a more general theory. There is no way to circumvent the "topological" restrictions within a formal theory; one has to "transcend" its limits intentionally referring to something beyond its reach and hence look at it from the outside.

In these lines, one could specifically treat the problem of *logical circularity*. If the geometry of the inference space were nearly plain, any case of a logical circularity would mean a logical fallacy. However, in a more complex theory, circular trajectories may indicate either the presence of singularities or an essentially non-Euclidean global topology. In the former case, following a circular trajectory around the singularity point would open new "branches" of the same inference space (like the branching of the logarithmic function in the complex plane). The sequence of transfinite ordinals provides yet another (nontrivial) example of that kind. The latter possibility can be readily observed in any dictionary defining one word is through another and the other way round. This kind of logic is also characteristic of the categorical schemes in philosophy; it lies in the very basis of scientific methodology.

Finally, one could touch the problem of *infinity*. For the extensive (spatial) paradigm, infinity is a datum, and it can only be postulated as a peculiarity of the inference space geometry. For the serial (temporal) paradigm, infinity is a process that can never end. The two kinds of infinity are respectively referred to as *actual* and *potential*. For some classes of "ergodic" theories, there is a correspondence between the two types of infinity, and a method of mutual conversion. For instance, any smooth dynamics is ergodic; alternatively, ergodicity may be found in certain modes of chaotic motion, while other nonlinearities produce various quasi-ergodic spaces with singularities. In such manifolds, one has to draw distinction between potential infinity and openness. Unreachability of a point in a finite way does not necessarily mean that the point is infinitely remote and cannot be reached at all; most probably, it just belongs to the (singular) border. The obvious duality of infinity and singularity reminds the interconnectedness of space in time in special (and general) relativity theory; the two kinds of infinity are thus understood as the different aspects of the same.

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