PHILOSOPHY OF MATHEMATICS

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These preliminary notes date back to 1984–2000. They were intended to become a part of an extended version of the already published draft of the *Philosophy of Consciousness* (Trafford 2009), as a kind of natural-scientific illustration. Here, a few relatively complete fragments are presented, just to illustrate the general direction of thought and to hint on the scientific background of the other expositions of unism. Each note is self-contained, and there is no particular order that would impart a sense of monograph to the whole.

Flames of Math

The history of mathematics is one of the longest histories of the world. Since the early days of humanity, formal inquiries were given proper names. Though the findings of Pythagoras (and the Pythagorean bravado of Plato) are still closer to mythology, the post-Aristotle time seems to entirely sit in treatise and document. Probably, this heap of data has something to do with the development of mathematics as such. Still, it is much more reachable from the philosophical angle, including both philosophy of math and philosophy of science in general. Once we need a historical allusion to support an abstract discourse, there it is, the illustrative mathematical fact. An encyclopedia of method, that's what we've got.

However, philosophy does not come to mere cognition theory. An exaggerated care for the techniques of knowledge mining is a good way for somebody to distract the public attention away from much more important problems. Accordingly, too much scientism in philosophy (as well as its formal inverse, phrase mongering) is nothing but a disguise, a showy trick. Real wisdom may look like science; but it does not scorn neither a vivid picture nor a piece of (non-abusive) exhortation. In any case, philosophy is to be primarily concerned with human deeds, examining their worldview and the view of the world's future.

To attenuate the traditional pretentiousness of mathematical philosophizing, let us commence with a flaming metaphor, tracing an analogy between the history of mathematics and the history of conquering fire. The both stories are of approximately the same size.

Primitive mathematical knowledge was mainly implemented in the daily practices of counting, measurement, distinguishing forms. It was preserved in a practical manner, passed from the tutor to an apprentice, from one generation to the next. Rare independent ploys were immediately incorporated to an appropriate craft chain; the chains ramified and replicated, so that the overall level of mathematical competence grew millennium be millennium, then century by century, then age by age. The history of fire shows exactly the same: we cannot yet produce fire, but we can keep it and pass to the others. The both techniques were insistently improved, which might keep it burning for a few thousand years.

Later, people learned to produce fire by friction. This liberated them from the need of centering the culture on a fireplace; a new hearth could be built anywhere at any urge. Fire became more reliable and more secure. As a result, the whole bulk of technologies related to fire conservation and transfer gives way to the progressive technologies of fire production. For this end, people invented ingenious aids and appliances, designed new surfaces with efficient friction control, used steadily flammable substrates, and so on. The hot motivation accelerates the development of industry in general, which even more simplifies fire management. Eventually, we come to a real masterpiece, the climax of the method, an ordinary match.

Similarly, in mathematics, the first clumsy substantiations get gradually polished up to becoming a well-regulated deductive scheme sometimes teachable even to robots. The numerous reformulations of the axiomatic carcass of mathematical theories provided us with a selection of elegant patterns to refine any new theory starting from just a few basic considerations.

Well, friction is not the only way to do fire. There is a parallel industry based on spark production at the impact of one solid against another. Here, too, one finds a lot of progress-boosting opportunities and comes to the modern lighter as an acme of the trade. Of course, the two branches interact, speeding up each other. Virtually, they use the same principle: we need to introduce a heated body in an easily flammable medium and to lead the combustion to a well-controllable mode. Which is the best? What is primary? A void question, since it is the practical considerations that rule. We use what better suites our needs. In the same manner, the agitated discussions about the foundations of mathematics do not much influence a working mathematician, who may resort to one approach or another to enhance the overall feasibility. Just aim at a stationary burning, that all.

In this way, the humanity approaches a new epoch in its fiery history. We learn that open fire is a sort of plasma, and the flames could be perfectly tamed when confined to a controlled gas flow or even put inside a sealed discharge tubes. As a first step, we develop a novel ignition technique, electrically incited discharge in the air or a special gas mixture. We no longer depend on the errant ways of tiny sparks; now, this is a full-fledged electric arc. We proceed with using electric lighters to kindle open fire; still, the principal direction of technological development is shifting elsewhere. The same trends could be observed in modern mathematics as well: the focus shifts from derivation and calculation to modeling. The mathematical fire is now playing in computer hardware, in algorithms and networking.

Don’t stop at that. We know about yet another (exotic) way of flame production, namely, by concentrating light from the sun. This method never occupied a significant place in the common trade, since the good lenses it needs came on the crest of the other ignition technologies; solar lighters have remained a children’s toy or a playful rite. On the other hand, the sun is here now, and hidden in the clouds the next moment; one cannot put in in the pocket like a box of matches or a gas lighter. There are regions where you need to wait for a few months to admire the sun over the horizon. So, can we get anywhere in these lines? Yes, we can. At least, we arrive at a big dream. Indeed, the Sun is the virtual source of all flames on Earth. Why not dare to light many artificial suns? After all, the Sun is a mere dwarf star in just one of the myriads of galaxies, while we are humans, and we can endeavor more than that, we’ll light new worlds.

Probably, some of the mathematical trifles that do not yet attract any academic interest keep the light of a mathematical dream burning within us, making us long for mysterious reigns beyond the formal universe of the present. It is much later that we’ll get really acquainted with them. For the time being, please, no disparaging remarks about philosophy! Yes, it is out of any rigor. But here, we are not entirely tied up by formal decency, and we can play and dream.

Mathematics as a Social Science

Most people (both mathematically versed and far from any math) would agree that mathematics is a science. Some of them, however, are apt to believe that this is a very special kind of science that lies in the basis of any science at all, providing the necessary pre-requisite for cognition as such. Finally, there are those who treat mathematics as an innate ability, the supreme knowledge embedded in the human animals in a mystical way, as a primordial touch of consciousness.

There are, indeed, serious reasons for all the above. Apparently, mathematicians behave like other scientists; at least, they talk the same way. On the other hand, it may be rather difficult to say what exactly they study in all those formal theories, whose practical significance finds recognition many decades (or even centuries) later, if ever. For an outer observer, mathematics is like a swarming ball of protoplasm that would suddenly sprout in one direction or another, to give birth to a new theoretical science, in the regular sense. To put it bluntly, a science must study something objective, that is, lying outside that very science and taken as an external source of facts and an application field. In this relative objectivity principle, we account for the whole hierarchy of indirect research, with higher-level sciences built upon a number of other sciences, however abstract, serving as a kind of empirical foundation. Like in any hierarchy, the “up” and “down” directions are readily interchangeable, and one could encounter the situations when two sciences lie in the empirical background of each other.

With mathematics, one cannot get rid of the impression of arbitrariness, inherent emptiness of any discussion, since one can easily develop a formally consistent theory starting from a collection of random assumptions, which are all equally acceptable, and there is no obvious reason for choice.

This was not that way in the ancient times, when mathematical knowledge came from immediate practical experience and satisfied people’s everyday needs. Social differentiation and division of labor have detached the skill from its applications; however, up to the end of the XIX century, there was a hope for a “natural” foundation of mathematics, the common root of all the further abstractions. Multiple geometries, the algebraic revolution and computers have dispersed that illusion, and now, we are left face to face with our strange ability of constructing imaginary worlds that no one will ever inhabit.

But did the character of mathematical knowledge really change? Despite all the formal games, the basic feeling of a number and a shape remains intact, while the alternative notions of rigor manifestly continue the same line of causal arrangement that has always inspired technological progress, from the troglodyte magic to the modern robotized industry. If we do so and so, we are bound to finish as expected, unless some external circumstance (including the operator’s blunders) abruptly modifies the operational environment.

This brings us back to the scientific status of mathematics. The object area of this peculiar science could be elicited using the hints from its early days, when its practical origin was yet evident enough.

Human activity is always aimed at producing some changes in the world. Each typical mode of such change gives impetus to developing a special science, with the object area related to the objective organization of the prototype activity. The hierarchy of sciences reflects the hierarchy of common activities. However, there is a fundamental distinction induced by the universal organization of any activity at all, which implies a conscious subject to take a portion of the world for an object and intentionally transform it into some product. As a result, each product can be characterized from two complementary aspects, as a kind of object (the material product) or as a representative of a certain way of action (the ideal product). This inner complexity of each product has eventually resulted in two complementary branches of science: the material aspect of activity is targeted by the so-called natural sciences, while knowledge about the modes of action is aggregated by social sciences. The latter name is appropriate since there is no individual that could exercise conscious activity outside any society at all, and the very idea of consciousness is only meaningful in the social context. That is, the subject of any activity is hierarchical, and any individuality belongs to that hierarchy along with the numerous forms of collective subject, from the family of two up to the humanity as a whole.

Now, does mathematical knowledge refer to any material things? No, it doesn’t. There are very few people who would consider mathematical constructs as self-contained things, existing on themselves (this philosophical position, known as objective idealism, is usually associated with the name of Plato). Intuitively, mathematical knowledge is rather about some common properties of things; but, for us, the only relevant properties are those that are significant for using things in our activity. Mathematics is, therefore, to study certain modes of human activity, and hence it must belong to the class of social sciences. In other words, mathematics brings us knowledge about ourselves, just as any other humanitarian research. This perfectly explains the apparent arbitrariness of mathematical theories, since social sciences take the world under a subjective angle, including the freedom of choice. In our everyday life, we have to decide on the appropriate modes of action; this, in particular, is reflected in the versatility of mathematics. Philosophical materialism, however, holds that no choice can be entirely arbitrary, and that the variety of available options is always determined by the objective organization of the world, by the nature of things. This circumstance is responsible for the apparent rigidity of the mathematical method complementing the apparent arbitrariness of the premises.

One could expect that new modes of mathematical thought would come in response to significant cultural shifts; however, the humanity has not yet (at least on the memory of civilizations) experienced revolutions of that scope, and we are still quite comfortable within the existing paradigms. Nevertheless, some hints to the open possibilities might be drawn out of the several methodological turnovers known in the history of mathematics. In any case, no science can ever reach the state of permanent completion; though some sciences (including certain mathematical theories) seem to have exhausted their creative impetus, their abandonment is of an entirely local significance, as there is always a chance of running into an interesting feature that has earlier been irrelevant, or just overlooked.

During the periods of relative stability, natural sciences develop an inner organization that drives then away from nature, to resemble the humanities in the very occurrence. A typical mathematical (or physical) paper is packed up with metaphors and allusions, using the regular language (indispensable in any discourse, however formal) in a very loose manner, mentioning thousands of names (which is intended, but fails, to reference earlier introduced ideas), lacking conceptual and theoretical consistency, as a clear exposition of the matter is impossible without a range of assumptions and preliminaries beyond any tractability. Professionalism gets almost entirely reduced to mastering the parochial slang, while logical transparence is sacrificed to erudition. Everything is made to impress the public rather than educate it.

A novice will find modern science almost incomprehensible, since an individual life is not enough to get just acquainted with all the parental work, nothing to say about a critical examination. Considering the chaotic character and inevitable circularity of references, there is practically no way to check the logical consistency and factual substantiation of any special report. The validity of reasoning is no longer a matter of proof, but rather a kind of common consent, prejudice or academic fashion, so that the whole of science virtually develops from one level of belief to another, rather than from truth to more truth. In this context, credulity and good memory are much more important for a student than inquisitiveness and prehension.

As a result, the overall structure of mathematical knowledge remains utterly conventional, just like in social disciplines similar to law, or accounting. The absence of a natural organization makes it impossible to establish a standard reference frame, to make mathematics searchable. This is a dictionary with no alphabet, sorted by random criteria, like keys and the number of strokes in Chinese and Japanese hieroglyphic dictionaries (or abstract hash values in computers). Eventually, there are too many characters to learn, and the whole thing splits into a number of traditional areas poorly communicating with each other.

Well, every cloud has a silver lining. In its chaotic mass, mathematics just cannot come too restrictive in natural sciences, leaving more room for metaphorical usage and losing the aura of magic that led many scientists to overestimating the role of formal manipulation and starting to toss phantasies instead of studying nature. Given the limited accessibility of advanced mathematical methods, we have to organize knowledge according to the structure of the objective area rather than stretch observations to an arbitrary formalism; this may reveal uncommon structures that could eventually push forward our mathematical thought.

Typically, a working scientist (*e.g.* in physics) has an individual mathematical toolkit, a store of standard components to reuse in any new theoretical model. However, when it comes to a drastic conceptual change, the already available forms are no longer sufficient, and a mathematical description has to be invented from scratch, since it is almost impossible to find the relevant pieces in the body of modern mathematics (counting out the always-possible random encounters). In case of success, mathematicians would assimilate some of such handicraft to the earlier introduced constructs, or add yet another *ad hoc* theory to the rest, to keep on with piling up formal junk. The portions of mathematics that penetrate other sciences are nothing but the coming back of their own inventions reformulated and "refined", spiced up with a scent of "rigor". This, again, resembles the situation with the humanities: to gain an official status, a new teaching needs authoritative support, a formal assignment and right to compete; later on, each authorized discipline can play itself a role of official authority for the newcomers. The pretense of mathematics to the absolute dominance in science is a neat replica of the superpower image of the USA on the political and economic stage. The inner discrepancies of the American society leave enough room for the other nations to break the dictate and develop on their own, thus influencing the development of the USA as well. Someday, science will probably abandon the idea of scientific ranks and forget interdisciplinary competition, to grow a new hierarchy of knowledge that would not distinguish natural sciences from anything "unnatural" or "supernatural". This must obviously follow the overall democratization of the world order, annihilating the market economy, class society and any kind of competition, throughout both the human culture and the world.

Mathematical Relativity

For many centuries, the power of mathematics has been associated with its deductive structure, so that many properties of a mathematical object could be formally derived from just a few fundamental statements. This circumstance tempts mathematicians into a common belief that the whole mathematics could be constructed as a deductive system, as soon as we fixed the most basic ideas. The search for the universal foundations of mathematics has never ended; a few general platforms have been largely explored, but neither of them came out to be satisfactory in all respects, and the question remains open up to now. Moreover, there are different schools that do not get along well enough; they seem to be irreducible to each other, leading to the kinds of mathematics that can intersect in most practically important domains, but do not coincide in full, with each theory producing a range of statements impossible in another approach. That is, instead of an all-embracing consistent formalism, we get a number of options, equally rigorous in their own sense. Once believed to give the ultimate truth, mathematics tends to dissolve in the numerous alternatives, and no mathematical truth is absolute nowadays. This makes mathematics a regular science like any other, rather than a common arbitrator and decisive authority. It may be bad news for those who came there for a bit of soul comfort in the uncomfortable world, but, no doubt, getting rid of blinding exceptionality will be useful for the development of the mathematical science proper.

In particular, we find that the paradigms of other sciences readily penetrate mathematical thought opening new promising directions of research. In fact, this is the principal way of development in mathematics, since any insider interests do not get beyond minor specifications and postponed proofs, while inventing a really valuable mathematical construct is impossible without a recourse to the current cultural demands (both in the material sphere and in all kinds of reflection). The job of a working mathematician is to attentively look at the world and seek for the ways of action common enough to admit of a kind of formalization; given the bulk of the already existing knowledge, such formalizations mostly adapt earlier theories, but sometimes, they give birth to an entirely new idea, and the lure of discovery is an essential complement to personal curiosity.

When it comes to the foundations of science, that is, the science's self-reflection, comparison with the others is of crucial importance. One cannot look at oneself but with the others' eyes. Here is a wide field for attractive metaphors, which gradually take the form of an intentional framework, and finally a unification platform.

On the topmost level, we are to explain why we need all those different paradigms together and outline the conditions for their interoperability. That is where the idea of relativity naturally enters. The possible unification platforms become the obviously analogs of the physical frames of reference, while the standard proofs of equivalence play the role of coordinate transforms. Just like in physics, there are families of dynamically different frames that cannot be reduced to each other without additional assumptions (inertial forces); however, within each family, all the relevant science is the same regardless of the choice of a particular formalism. Just like in physics, we are facing the problem of objectivity, since distinguishing real topics to discuss from spurious issues due to an inappropriate model is a non-trivial task requiring a wider context to embrace the whole hierarchy of such distinctions, allowing for hierarchical conversion and hierarchical development. Each individual layered structure can become a mathematical theory; but there is no universal mathematics equally applicable to any conceptual frame at all.

The impressive efficiency of science is in its power of abstraction, and hence lack of universality. Still, there is no abstraction without something to abstract from, and hence any science (including mathematics) can only approach the integrity of its object from one side or another, never capable of any exhaustive description; even combining all the existing and possible sciences together, we still remain abstract and approximate, since, to be truly concrete, one needs to advance beyond science, into the practical sphere. And here is the explanation for the frame-of-reference mystery. We are never interested in the particulars of any individual science; we only need schemes of action (prescriptions, recipes) applicable within a range of common situations. Of course, such schemes cannot be of an absolute value, they essentially depend on the modes of activity. As long as different people do something in a similar manner, they share a collection of formal implements, differently arranged in each individual reflective household. This objective structure of activity underlies the notion of a reference frame (or a conceptual platform).

The universality of reason means, in particular, that any aspect of activity can become a separate activity, and any separate activity can be included in another activity as its specific aspect. This is how scientific reflection becomes institutionalized science, and one science can be used as a paradigm for another. However reflexive development of mathematics does not mean that we could seek for the grounds of mathematics in mathematics itself, just laying out the foundations of science in terms of that very science. Mathematics is hierarchical; it cannot be reduced to a trivial flat structure. There are different mathematical theories, the abstract models of the corresponding objects; a mathematical theory of another theory is essentially different from its target, they belong to the different levels of hierarchy. Within a definite hierarchical structure, we can find that many theories are basically isomorphic to each other; still, isomorphism does not mean identity, it could rather be pictured as *identity in a relative way*, within a higher-level integrity and in this particular respect. This closely resembles physical frames of reference related to each other by inertial transforms (isomorphism) which are considered to be “physically equivalent”. There is an obvious logical circularity, as we have to refer to physical equivalence to define inertial systems, while the notion of inertial motion refers to a class of physical commonalities; this circularity can only be broken from a higher-level perspective (the practical context). Similarly, seeking for the foundations of mathematics, we only establish a common practical context, allowing for formal transitions from one theory to another within the same higher-level paradigm. But we never invent this paradigm from nothing; it must reflect the common ways of action, and any change of paradigm will always come as a result of a cultural shift, virtually due to the development of the mode of material production.

Mathematics and Computers

One can often hear that the computer revolution on the boundary of the XX and XXI centuries was due to the achievements of modern mathematics, and, in particular, that the major trends of software development merely implement some abstract mathematical ideas. Despite of all the apparent evidence, including the memories of the great programmers, this viewpoint can hardly be accepted without reserve. There is no reason to trust the subjective accounts of IT gurus more than the feelings of any other person. People rarely pay attention to the hidden motives of their activity; all they can report is a number of transitory goals, a partial representation of the cultural background they normally don’t observe (as the attempts to shift the focus of attention to the conditions of one’s activity will terminate that very activity and unfold the activity of reflection). When a programmer insists that some product was inspired by a beautiful mathematical theory, this is but a superficial impression, while the real source of inspiration is to be sought for elsewhere. Quite often, mathematical considerations are added in retrospect to an already available practical scheme, as a kind of justification or promotion, and, indeed, no computer program exactly follows its mathematical “prototype”. Programming is a practical discipline that cannot be reduced to pure mathematics, even in the guise of “computer science”.

The very words ‘computation’, ‘operation’ and ‘algorithm’ are said to come from mathematics, and this is admittedly an argument in favor of the primary role of mathematical science in computing. Even assuming the validity of such attributions, terminological history has nothing to do with the origin of the corresponding notions; the same notion can be expressed in quite different terms, or go without any verbalization at all. But actually, the tales about the mathematical roots of the fundamental concepts related to computing are mere fantasy. At a closer examination, one would rather suspect exactly the contrary: mathematicians invent symbolic notation for something that has already become a common practice, a part of our cultural experience and, in a way, our everyday life. Long before any formal “proof”, people used to convince each other by the very way of action, and it is only much later that the typical “proofs” have been systematized and codified under the name of formal logic, which, in a couple of millennia, has been further truncated to what we know as mathematical logic. But if you scoop up a little water from a well to slake your thirst, the well is still there, regardless of your scoop of water, and the folks may use it in many other ways. Some alternative notions of proof have already been assimilated by the mathematical thought; many more are yet to be discovered, without any diminishing the importance of whatever already known. Similarly, the idea of an algorithm, a formal prescription for the solution of a class of similar problems, has existed in human culture from the most ancient times, long before the first sprouts of mathematical science. In the human society (and in higher animals), any successful action tends to promptly become a model for other actions; primitive people did not clearly see the reasons of efficiency, and they had to stick to superficial details, fixing the form of action regardless of its real content; this rigid construction was then imposed on all the members of the community by any authorities and sanctified by the priests, thus leading to a social norm, a common pattern later reflected in the arts, science and philosophy. These formal prescriptions were absolutely necessary on the earlier stages of human development; they play an important role in the modern culture as well, supporting its stability and the congruity of its evolution. However, a creative person would not exaggerate such formalities, however productive. They are always restricted to specific social conditions, a certain level of cultural development; a slightest novelty inevitably breaks the rules. That is why mathematics has never followed the algorithmic line in its own development, and even less can it advise it to programming.

Step-by-step instructions exist for many practical activities far from mathematics. Take any drawing class, or a dance school, for an example from the arts; a quick user guide is available for any home appliance, and there is no professional training without learning a number of routine operation sequences. However, the algorithmic component can never prevail in ordinary life, which is full of unexpected turns demanding an immediate creative response. It is only simple artificial objects like mathematical constructs that extensively allow formal operation; any real object is much more complex than the most intricate mathematical phantasm. Still, in many practical cases, we can control the level of relevance, monitoring just a few principal traits and compensating the rest as side effects. This is what we explicitly do in programming.

The power of mathematics comes from oversimplification. It prompts us to adjust our activities to match the level of simplicity marked by formal mathematical constructs thus making the whole thing simpler and better tractable; of course, the actual complexity does not go away, but it can effectively be pushed into the background, to the lower levels of hierarchy. That is, the practical value of mathematics is to hierarchically structure our everyday life, which opens new perspectives for efficient algorithmic procedures, which suggest more abstractions, and thus *ad infinitum*. Here, we are facing the old issue of the egg and the hen. Cultural progress gets reflected in mathematical theories, which, in their turn, stimulate certain cultural changes. Such circularity is a characteristic feature of any development at all, and it may seem that the idea of a special role of mathematics thus gains additional points. Well, in a way, this is a valid description of an old-style theoretician, who would not care for anything beyond fundamental science (as long as there is no issue of procuring the required material support). However, cultural progress is currently getting so fast that there is no time to develop a solid mathematical background for any operational regularity, and one has to either proceed on the trial-and-error basis, or adapt some existing mathematics, being fully aware of the transient and approximate nature of such *ad* *hoc* schemes. In this mutable world, mathematics is expected to provide principles rather than solutions, thus coming much closer to social science, with practically nothing to compute. Similarly, computers have evolved from mere computing machines towards universal control devices, with any numerical calculations mainly reserved to the presentation level. In this respect, modern computers are much closer to the ancient mechanical and hydraulic toys, which have later passed their experience to the industrial automated production lines.

The idea of a computing device came as a natural continuation of all the preceding technological development, as rational reasoning was getting ever more formalized within the first advances of the institutionalized science of the modern type. This algorithmic approach to human creativity was promoted in the early Utopian writings, and it grew very popular in the arts long before any scientific applications. Just recall the well-known dice-driven musical composition algorithm by Mozart, as well similar schemes by his contemporaries and predecessors. However, the Pythagorean tradition of treating mathematics as a kind of art has later been reversed, and many artists were apt to believe that the inner integrity and harmony of the arts is due to some mathematical laws *a priori* built in the human nature.

Anyway, automating the routine operations of a mathematically laden scientist of the XIX–XX centuries was quite an obvious suggestion, given the habitual presence of manual “computers” like abacus and mechanical adding machines. All one needed to do was to supply a mechanical device with an independent source of power and feed it a program in a manner well habitual in mechanical pianos or textile industry. The very fact that the practical implementation came rather late indicates the auxiliary role of mathematics (and formal science in general) in the history of computers; the progress was primarily due to hardware development, new processing technologies rather than processing rules. And it is these empirically found operation modes that shaped mathematical programming, not the other way.

Though modern computer science is a brainchild of the “digital revolution”, analog computing has never lost its practical importance and conceptual significance. Discrete mathematics is a relatively narrow branch of mathematics in general, while the idea of approximation rests in the core of any science at all. Both ways are equally productive: discrete algorithms are often approximated by some smooth data flow, as well as continuous processes get discretized in their digital models; computer modelling is absolutely dominating over numerical calculations in modern science and industry. Neither of the opposites can live without the other. To listen to digital music, we need an analog device, while discretized music is much more tractable and safer to store. In this context, the traditional musical notation and the tradition of creative performance could be considered as a prototype of modern information technologies. Eventually, any computer is an analog device, albeit used in a digital manner.

Here is where we come to the point. Mathematics is essentially static; it studies the structural aspects of human activity. On the contrary, computing is all about time; without intentionally ordering events, one just could not speak about computation (operation). To bridge the gap, one needs to somehow introduce time in mathematics and structure in programming. However, mathematical time cannot be but yet another structure; similarly, programmatic structures can only be meaningful in a dynamic sense, as operation types. That is, as soon as we have established any correspondence between mathematics and computing, it must inevitably be broken, to start a new cycle of reflection.

To illustrate this, let us turn to the procedures of measurement. There is an important difference between space and time. Spatial dimensions are static (structural), and we need to deliberately move in space, to be able to evaluate any distances. Not so with time. Moreover, to accurately measure time, we need to stay as still as possible, to avoid the influence of any spatial displacements onto the indications of the clock (possibly, in the form of gnomon, hourglass, or clepsydra, or even a calendar, for longer periods). In a way, this is the very meaning of the words ‘space’ and ‘time’: space is what can be taken at the same time; time is what happens in the same place. No relativity considerations will cancel this fundamental distinction, as they can only refer to the specific numerical representations (modes of measurement). Considering the relativistic interval, rather than the separate measures of space and time, is merely a reformulation of the same trivial statement: moving in space, we introduce certain errors in time measurement, while lack of simultaneity results in slightly distorted spatial relations.

However, as soon as we choose a particular device to measure time, we introduce a specific structure, a temporal scale. All the local events can then be related to the marks on this scale, *as if* they were taken simultaneously like the points of some space. We can formally combine this scale with the former spatial dimensions and study the geometry of the resulting space. But this immediately poses the problem of comparison, with the evolution of thus obtained geometrical structure producing some other time, which would produce a different time scale, and so on. Both space and time become hierarchical; this hierarchy can be represented with many hierarchical structures, depending on the chosen modes of measurement.

A margin note: in the early science, any time scale implied a spatial implementation in the form of a dial-plate; that is, the moments of time were labeled with some spatial positions. The exact shape of the dial does not matter; just for one example, take the usual watch face with a few turning hands related to the different time scales (in this particular case, one time scale can, in a way, be reduced to another; this is not generally so). Today, time indication can be rather sophisticated: instead of single spatial positions, time marks may involve various spatial distributions (by definition, taken at the same time). Still, in any case, a time mark is a virtual activity developing “in no time” on the current scale; namely, the process of measurement (mark-up).

Now, with computing (understood in the most general sense as a series of data control operations), the situation is exactly the same. At every step, one structure is to produce another; however, without a particular time scale, there is no way to define any structures at all, since the “simultaneous” events of one level may unfold in temporal sequences on another, and *vice versa*. To structure computing (for example, to distinguish the initial data, processing and the result), one needs to compare it with some other process (benchmarking). Any structure is only meaningful within a higher-level structure; any sequencing implies folded lower-level activities. The hierarchical organization of human activity is reflected both in mathematics and information technologies, as well as in any other cultural area.

Mathematical Measurement

Mathematicians often picture their science as entirely abstract and unrelated to reality. They hold that the development of mathematics follows its own ways, free from any practical needs, with no concern for the possible applications. Ultimately, the principles of mathematical reasoning are deemed to be eternal and non-mutable, prescribed once and forever by no matter who. Inebriated by the obvious success of formal methods in science and engineering, mathematicians are apt to believe that their science is to provide supreme (absolute) knowledge and the final criteria of consistency and truth.

Such is the body of common delusions about mathematics shared by many people who are not trained enough to observe their utter absurdity. Wisdom is difficult to learn, it has to be gradually acquired through persistent effort throughout one’s life; studies like that are not much in favor today, as the leading economies are based on the division of labor. Professional philosophers lack scientific education; their chancy erudition can hardly become the right soil to raise universal ideas; moreover, modern professionals have often to be engaged in anything but their professional activities. The common public has almost no opportunity (and is not intended, nor encouraged to go in) for extensive reading; people remain widely ignorant, and they have no other choice as to trust to those who look authoritative enough. Thus, the present authorities feed us the authority of math.

However, a closer look at the activity of a mathematician will immediately reveal its overall resemblance to any other science, to the degree of a close kinship. Indeed, given a number of properties common to a class of objects, one normally tries to establish a number of interdependences, considering the possibility of prediction on the basis of a few already established facts. This is exactly what scientists do in physics, in chemistry, in biology, in linguistics, in psychology, or in history. This also pertains to learning, acquiring professional skills in any specific area, like medicine, storytelling, or plumbing. In any case, the regularities thus discovered are in no way arbitrary, they reflect the real order of things and the current modes of their usage. The only difference is in the nature of the objects and the domain of applicability.

That is, to comprehend the nature of mathematics, we need to grasp at least some idea of its object area. The details might come in the course of further study; but the general direction is to be chosen from the very beginning; otherwise, the activity just could not start.

Of course, indicating the object of mathematics is a nontrivial task, and hence all the controversy about the foundations of mathematics, and the range of common illusions. The situation is aggravated by the fact that mathematical ideas have never stopped developing, both in their wording and in their content, and it may be difficult to trace the origin of the currently recognizable fundamental blocks back to the roots of the trade.

Thus, today, we understand that there is something in common between 3 apples and 3 stones. With a little more mental effort we can admit that a collection of 2 stones and 1 apple is, in a way, like 3 stones, or 3 apples, and so is the collection of 1 stone and 2 apples, or even 1 stone, 1 apple, and 1 bird. With more experience in abstraction, we can also discover that 3 days, or 3 wishes, somehow fall in the same category. In practice, such associations are established through comparison of various collections of things (or just ideas) with the same reference collection (for example, 3 fingers). The final step is to remove (abstract from) any reference collection at all and speak about the *number* 3 as a common characteristic of any group of 3 distinct entities, regardless of their distinctions. This approach, stressing the universal commonality, is called *quantitative*.

Alternatively, one could focus on the very act of distinction, the individuality of things, treating them as *qualitatively* different, and hence incomparable. Obviously, this absolute distinction means as absolute equality, since the very impossibility of comparison prevents us from telling the difference as well as from establishing any commonality. That is, we can count unique things as indistinguishable, and hence identical. This gives us yet another mathematical primitive, an *item*, just something to count (a point, an element, an operation, a link, a value). In particular, we can count numbers.

The quantitative and qualitative aspects are intrinsically related to each other, they always go together in any practical act. Speaking of commonality, we mean distinction; speaking of distinction, we mean identity. We consider quantitative grades within the same quality, as well as the quantitative limits pertaining to a specific quality. This means that every individual thing must also be characterized in yet another way, allowing us to distinguish quality from quantity and put them in the same context. It is only in respect to this common base that some aspects of the whole could be called qualitative, while some other aspects would provide a qualitative assessment. In philosophy, this unity of quality and quantity is known as *measure* (not to confuse with the narrow mathematical term). Assigning something a measure is an act of *measurement*, in the most general sense.

For measurement, it is important that different things can be considered as equivalent, to a certain degree, that is, commeasurable. This possibility is primarily related to the very nature of human activity, which always takes some object to produce some product. The product could be called a material implementation (realization) of measure, since any objects within the same activity become comparable in respect to its product: they are either fit for production, or irrelevant, with the whole hierarchy of the possible grades of applicability.

Quality and quantity are the two complementary aspects of measure, and hence their distinction is relative. The same measure may compare things in a different respect, without changing the sense of comparison. This observation is almost trivial, since the result of any measurement (quantity) entirely depends on its method (quality). The same work can be done in many ways. Depending on how we distinguish countable things, we will obtain different counts.

The universality of the subject means that every two distinct things in the world can virtually be compared; however, any comparison needs an appropriate measure, which has to be established in practical activity, and hence the scope of actual comparability is determined by the current level of cultural development. That is, the very thought of comparison already means that there is an appropriate social background, and no idea can just enter one's head without a cultural instigation. Of course, there is no need to track any mathematical construct at all back to material production, since every activity, within a cultural environment, can become an industry on its own, representing some higher-level cultural regularities (reflexive activity). In particular, the product performing the role of a measure does not need to be a palpable thing; sometimes, it may be just an intricate interrelation almost impossible to embody or express. Nevertheless, the cultural determination of formal operations cannot be eliminated, even in the most abstract areas of science.

Within the same measure, the objects in the domain of commensurability are, in a sense, interchangeable, just like trade articles in the market. Each of them can be taken for a *unit*, with the other objects somehow related to it. This specific implementation of measurement is certainly not unique, since any other object can serve as a unit as well, producing its own *scale*, so that anything could be associated with a definite *grade* of the scale, which provides a common measure for all the objects of the same degree. Any measure will thus produce a *hierarchy* of scales, which is extremely mutable and multiform, each particular representation being a hierarchical structure, virtually corresponding to a possible structure of the matrix activity. Such hierarchical structures are what could roughly be taken for the object of mathematics as a science. Each branch of mathematics refers to a class of typical structures, the different ways to organize people's activities.

As an area of activity, mathematical science can be structured in the same way; the ideas of an exceptional character of mathematical knowledge are largely due to this apparent reflexivity. Still, mathematics is not unique in that. Any science at all is reflexive, since its cultural existence is impossible without the feedback from its application area, and many sciences involve the processes they are going to describe. Can you imagine a physicist that would not be a physical system? Or a biologist that would not be an organism (or a community of organisms)? Economical science is obviously engaged in a sort of trade; psychology requires intellect and emotions; linguists communicate in many languages; geology develops on the planet Earth. The only difference is in the character of reflexivity: in mathematics, it often (but not always) becomes explicit and intentional.

In this context, one could picture measure as an instance of categorization: a scale provides a number of categories (grades of the scale), and each individual act of measurement is to put the result in a definite category. This is a most common operation in people's everyday life, and we don't even pay attention to its essentially cultural nature, the necessity of an embedding activity. Categorization is never trivial. It must historically (practically) form as a cultural pattern (a hierarchy of scales) on the basis of mere comparison (distinction) in respect to the product of some activity. This is the link between humans and animals; the latter can immediately assess the biological importance of a stimulus and take reflective action, but they do not develop any scales, except in hierarchically organized communities with dynamic function exchange (the prototypes of the human society). In other words, a category must objectively exist before we put anything in it; and this cannot be but social (ideal) existence. As soon as a category becomes "wired" in the decision maker, there is no real choice, and we can only speak about categorization in the metaphorical manner. The same holds for acquired "categories" established in individual learning and encoded in one's mental activity and neural patterns; however social by their origin, such "psychological appliances" do not much differ from bodily organs, and their rigidity is felt as a lack of responsibility and limited freedom. From the hierarchical viewpoint, this means too much preference for one hierarchical structure to the detriment of the others, the suspension of hierarchical conversion.

It is important that measurement (elementary or discriminative) is not indispensable in human activity, and, in many cases, people can avoid it. There are other modes of assessment, and the integrity of the human culture is only achieved in the interplay of all the possibilities. Thus, to tell whether all the members of the family have come for dinner, we don't need to count them; we can detect somebody's absence at a glance. Similarly, we don't need to count the buttons on a coat to find that one is missing. The shape of the pillow is of no importance as long as it does not disturb our sleep. And we don't care for the truth of somebody's words as long as our communication is only to confirm sympathy or disgust. Some ethnic groups never come to the general ideas of number, or shape, since they don't need them in any practical respect. Similarly, the quantitative aspects of reality are irrelevant to little children below some culturally determined age. Even in science, a bulk of factual or operational knowledge is primarily accumulated without too much bothering about formal distinctions; this is, for instance, how the method of contemporary mathematics has virtually been born. However, the philosophical categories 'quality', 'quantity' and 'measure' reflect the fundamental organization of the world (regardless of conscious activity) and are universally applicable, so that anything at all can (thought does not need to) be measured (compared, categorized, evaluated), in complement to all the other attitudes.

I dwell so much on the preliminaries, since the details of mathematical measurement can easily be recollected as soon as there is an understanding of their cultural determination. Any mathematician could do it almost in no time; a less trained person would additionally need refreshing one's school reminiscences. A working scientist is the most difficult person to persuade, just because of the many formal habits that have penetrated the very core of one's personality. So, let such people do what they are made for without too much caring about the foundations of mathematics, which would not certainly be the best application of their talents.

There is no restriction as to the type of scale to produce a mathematical measure. Any activity can produce various formal counterparts, depending on the way we unfold its hierarchy. The most fundamental mathematical notions are as subject to reassessment as any auxiliaries. In the above example with abstract numbers, one can easily observe numerous conceptual vulnerabilities. Thus, putting all kinds of items in the same row, we refer to some culturally established operation of counting; the measurement procedure involves labeling individual items with the marks on a standard scale (a number of reference objects taken in a fixed order). That is, counting assumes enumeration. Natural numbers as a mathematical construct provide an abstract scale that does not depend on the particular implementation; however, in any practical measurement, we have to choose an appropriate instantiation of that abstract scale, from fingers and abacus balls to neural patterns in the brain, people in a queue, or a programmatic iterator. No implementation is perfect, and it took the humanity quite a lot to come to relatively stable procedure of counting; however, no one can guarantee that these habitual operations won't fail in some exotic conditions, requiring a different method of measurement. Just think about the very common finger scale and admit that some people might have a different number of fingers, or no fingers at all.

Assuming the adequacy of the materialized scale (the counting instrument), we still face the problem of the order of counting, that is, the necessity to arrange the objects to count in a row, so that we could sequentially associate them with the marks of the scale, eventually exhausting the collection and coming to the latest scale mark, which is exactly what we need, the number of items. Normally, we don't much bother about that, since, in many practical cases, the result does not depend on the counting order. But, in general, we need yet another iterator, taking the objects from the counted collection one by one for us to be able to relate the next object to the next grade of the scale. This enumeration is not trivial; the result may depend on the sequence obtained. For instance, the members of the sequence can cling to each other if taken in a particular order, while showing no interaction in other arrangements. Take the sequences of letters that may or may not form words during measurements; if we (for some reasons) detect words as single entities, the outcome of measurement will essentially depend on the mode of enumeration. Yet another common possibility is related to finite objects that disintegrate after an objectively determined time interval (*e.g.* like in radioactive decay). If we count starting from the long-lived items, we risk overlooking their shorter-lived companions in the whole.

Continuing the time theme, one could observe that the result of the measurement also depends on the rate of enumeration, both for the item collection and the scale. If the item sequence is produced much faster than the typical scaling time, some items just cannot be counted. In the worst case, the scaling time may depend on the next item, since some items might be more difficult to cope due to their size, weight, mobility, or cultural dependencies. Alternatively, the scale might occasionally fire "spikes" in packets, so that the same item would be counted several times (which may, for instance, correspond to yet another mathematical primitive, a set with repeated elements, a "bag"). The "classical" activity of counting is therefore to proceed in an "adiabatic" manner, with enough time between successive counts for the two iterators (the item collection and the scale) to relax and restore the "standard" state before each step. Everybody who has ever had experience in computer programming (especially networking) knows that such a steady operation may often be not easy to achieve.

We naturally come to the conclusion that the mathematical notion of a number reflects a very special way of operation subject to numerous restrictions. However common in our current cultural environment, such operations may be utterly impossible in other cultures (or on the other levels of culture), where the very idea of a number would be inappropriate. Of course, this does not mean the impossibility of mathematics in general; some other mathematical measures might come quite handy.

A mathematician could object that their science should not care for feasibility, and mathematical theories could develop regardless of any application, just to explore the formal dependencies. However, mathematics is just a kind of activity, and it is bound to run into the same problems as anything else. The famous Gödel theorems provide a bright example of inappropriate arithmetic coding producing the illusion of commensurability where there is none. There are reasons to believe that any diagonal proof is, in fact, an indication of the inapplicability of the corresponding mathematical theory.

Just for illustration, let us look closer at the common distinction between ordinal and cardinal numbers. As indicated above, counting implies sequencing, and hence the numeric scale is used in the "ordinal" sense, as an order imposed on the item collection. Provided different orderings are irrelevant (and, in particular, they give the same item count), we could think of the number of items as a cardinal number, a characteristic of the collection as a whole, its "volume" (or "mass"). Formally expressed by the same mathematical constructs (numbers), such "bulk" measures reflect a different approach to measurement that does not need resolving individual elements within the whole. Instead of counting bricks in a pile, we could just weigh the pile and thus estimate the number of bricks in an indirect way, knowing the weight of a brick. Similarly, we could measure the length of the border built of the bricks from the pile; and, again, the number of bricks can be formally estimated knowing the size of a brick. Obviously, a bulk measure does not necessarily allow quantitative judgment on the elements of the collection; thus, in a heap of stones (instead of standard bricks), individual stones may be very different in size or mass, so that the weight or dimensions of the heap do not provide any information on its structure.

In physics, we also distinguish intensive (like temperature, pressure, or entropy) and extensive (like mass and volume) quantities: the former are evaluated for the whole system; the latter sum up from the corresponding values for its parts. There are obvious analogs of intensive measures in mathematics: the dimensionality of a space, the topological index, the momenta of a statistical distribution, or algorithmic complexity. Even if we can subdivide a mathematical construct into a series of smaller constructs of the same kind, the estimates of intensive quantities for the parts won't trivially sum up into the corresponding value for the whole. For instance, if we split a rectangle into several (non-intersecting) rectangles, the ratios of the side lengths for individual components have nothing in common with the same ratio for the original (compare with the areas of the parts that sum up to the whole area); one could also find intermediate cases: thus, the sum of the perimeters of the components rectangles does not equal the perimeter of the whole, but the two values can be correlated for some types of partitioning.

Intensive characteristics are related to the mathematical idea of a *shape*, as complementary to that of a number. The same considerations about identity and distinction apply here. One could roughly describe things as "round", "square", "jagged" *etc*. We distinguish smoothness from raggedness, chaos from order, similarity from distinction, or failure from success… In all such cases, we take different things and treat (employ) them in a similar manner. This makes these things virtually identical (and hence subject to quantitative description), but also different from other things that cannot be used the same way.

While a number is to stress the quantitative aspects of measure, shape mainly refers to quality. We can describe shapes with numbers; but such descriptions do not convey the integrity of shape, they only explain why we perceive certain things as different shapes. In many practical situation (in the zones of cultural stability), we can guess shapes from a selection of numerical parameters. However, this does not reduce shapes to numbers, since the same shape could arise under quite different conditions, where typical numerical expressions are no longer applicable. Conversely, the same collection of numbers may refer to an entirely different shape; we can visualize one shape with another, but this won't make them qualitatively equivalent. In other words, a shape is a higher-order entity respective to any particular parametrizations, and it is defined respect to other shapes rather than by any numerical expressions. On the other hand, a numerically parametrized shape can be considered as more abstract, since, among all the characteristics of the shape, it selects just a few; one could call such a parametrization “the shape of a shape”. This is quite similar to how we represent a spatial point with its coordinates, thus making it an abstraction of a point.

Shapes do not exhaust the range of possible qualities. For instance, we feel certain commonality between two apples that makes them not exactly like, say, two hedgehogs. Moreover, there are different sorts of apples, and even apples of the same sort may be of different grades. Each individual thing is virtually different from any other thing, and that is why we can speak of any individuality at all. Still, within a definite measure, its qualitative aspect will characterize the commonality of individual things making them countable units. One could consider shapes as *formal* qualities (a “quantitative” quality) abstracted from any particular measure, just like a number as an abstraction of quantity. In this formal sense, apples and hedgehogs could be put in the same category (of approximately round things) and counted on the same footing. This is where mathematics is at its best.

Shapes and numbers could be compared to space and time in physics. In the same way, the items composing a shape can be ordered (counted) in a kind of trajectory, and conversely, a class of trajectories can be associated with a definite duration (a number). There is the same mutuality, the necessity of distinction and its inevitably relative character. And just like time is related to cyclic reproduction of the world (or any its part), the possibility of counting (and any numeric evaluation in general) is due to the repetitive actions and operations in human activity. In a way, mathematics could be called a model of cultural space-time, as a level of space and time in general.

Associating intensive measures with shapes, we come to a very general idea of shape applicable to abstract entities as well. For instance, mathematical theories (and their elements) can be related to some measure of truth, which is obviously an intensive parameter, so that the truth of the individual statements does not imply the truth of the whole theory. Here, exactly like in physics, the same measure is applied to the whole in a sense different from measuring its components, thus unfolding a hierarchy of measure. In particular, the hierarchy of truth determines the shape of a theory. Of course, the same theory can be related to other intensive measures (like decidability, productivity, predictive power *etc*.) and may have a different shape from yet another perspective. These “special” shapes are not entirely independent. In some cases, intensive parameters may become extensive, and *vice versa*. Thus, the mass of a compound particle builds up from the masses of individual components in nonrelativistic macroscopic systems; however, the mass of an atomic nucleus cannot be reduced to the masses of individual nucleons. A complex system may behave as a whole in one context, while undergoing mere shape changes in another. Moreover, the sequence of changes may essentially depend on the system’s history. Similarly, a mathematical theory may link intensive measures to each other, thus making them “less intensive”. In the context of some theory, two statements (assessed as true or false) can be combined in a compound statement (using some logical junctions), so that the truth value of the result could be derived from the truth values of the components. Different sets of junctions differently shape the theory (producing its equivalent formulations), and one of the primary concerns of a mathematician is to determine the range of constructs preserving the same overall structure, which is indeed not evident from the very beginning, being revealed in the course of development, like in other sciences.

The Diagonal Fallacy

The formality of mathematical knowledge is both its principal advantage and the major source of conceptual problems. While a mathematician is engaged in studying a particular mathematical object, he may (like in any other science) never worry about the correctness of the typical operations. Still, any attempt to formally describe the very formalism, to scientifically study the method of science, is bound to fail from the very beginning. Such a formalization will necessarily be incomplete, since it pays attention to a specific aspect of the scientist’s work, while abstracting from the numerous informal procedures, which only can render science meaningful.

The foundations of mathematics lie beyond mathematics, and no metatheories can help. This is just another level of reflection, which is entirely outside the scope of science as such. No doubt, every activity involves a number of formal components that could be squeezed into scientific standards, so that everybody could learn them. In a way, all science is a factory of technologies, typical schemes of activity that can be reproduced in very different situations, including the development of science. The scientific product is a kind of huge textbook, a cookbook containing millions of life hacks, to conveniently rule out most everyday problems. In this respect, mathematics is in no way different from chemistry, political economy, anatomy, choreography or rose planting.

The discrimination of a formal structure in a real activity assumes a certain level of abstraction dividing all the aspects of reality into those that are relevant and those that are not. Such levels are not arbitrary in the history of culture, though a general regularity does not remove variations of any kind (which, in their turn, form a hierarchy of formality or meaningfulness of their own). It is important that different levels exploit different formal features, and there is no final, frozen knowledge, since (at least) new representations of the same are always possible, and we need to adjust the well-established theories to unexpected practical needs.

Formal knowledge is only viable within a single level of hierarchy. When a mathematician tries to formally join several distinct levels, this comes up as a contradiction.

The famous Gödel theorems provide a popular example. They have been given a lot of formulations, sometimes very far from the original arithmetic approach. A common reader will find these theories too technical, as they are oversaturated by minute detail and highly suspicious artificial constructions *ad hoc*. This resembles too much the machinery of a circus magician designed to distract the attention of the public from the insides of the trick.

Still, the affair is quite transparent. Any formal system can be represented with a number of statements[1] somehow assigned with the logical value "true" or "false". If some statement happens to receive the both marks, the formal system is called contradictory. But this only refers to the structural level. To make a system, we need to specify its input and output, as well as the method of obtaining the result from initial data according to a number of rules that constitute the inner structure of the system. The structure of a formal system is determined by the derivation rules to obtain a true statement from a number of other true statements. Of course, one can apply the same rules to false statements, but this will tell nothing about the truth of the result. In this way, any formal system is enhanced with the notion of *deducibility*, and we have to investigate its relation to *truth valuation*. In particular, a formal system is called complete, when every true statement is deductible.

Obviously, deducibility is like higher-level truth, the next level of hierarchy, known in dialectical logic as the negation of negation. Unlike the immediately true statements, deductible statements are not only true by themselves, but primarily, true *by derivation*. In practice, it is always possible to either make a particular statement an axiom or derive it from other axioms; these are the different ways to unfold the same formal system, thus producing its hierarchical *positions*. In any case, truth and deducibility belong to different levels, and their mixture within the same context would be an instance of inconsistency, a fallacy. Now, this is just what all the forms of the Gödel theorem attempt to do!

In the core of the trick, we find the so-called diagonal principle: a statement gets separated from its content, from the object area, and is declared to apply to anything at all, including itself. For a few millennia, logicians reproduce the same old liar paradox, adding ever more interpretations in the popular literature.

The proof of the Gödel theorem (in any formulation) involves artificially constructing a reflexive statement meaning: "every deductible statement is false". If this statement is deductible, then it is both true and false, and the formal systems is contradictory. But, if it is true, it is not deductible and the formal system is incomplete.

[1] I use the collective name of "statement", meaning all the assertive phrases fixing an element of knowledge, rather than conveying a question, a doubt, an instigation *etc*. That is, here, the term "statement" is taken in a rather broad sense. However, it is still much narrower than "saying in general". In this context, statements assume logical judgement, regardless of whether it is to be somehow substantiated or not.

That does it. All one is to clarify is how the conjurers fool the happy public.

For this purpose, mathematics (following the lines of philosophical positivism) employs a standard trick: the content of a statement is identified with the form of its expression. A sane person will hardly ever mean it. When we say, “the egg was boiled during three minutes”, we mean that very egg, and the boiling water, where it spent three minutes according to the kitchen timer. But never the sequence of words used to tell that.

A mathematician acts in the contrary manner: he declares that any formal statement exists insofar as it can be expressed by means of some formal language, and all the statements can be enumerated as soon as we have fixed the alphabet. The possible translation from one language into another is then reduced to mere substitution of the original characters for new ones according to a definite rule, which does not deny the very possibility of enumeration.

Sounds convincing. The public is enchanted with the artful movements of the magician explicitly constructing the language and formulating the theory, which is to eventually bring in the desired diagonal formula. Apparently, the Gödel theorem is an elegant mathematical result, a revelation of supreme science.

With all that, it is exactly science that it lacks. Science is always object bound, it studies something real and never asserts anything in general, but solely within its application area. If we start paying attention to the mode of producing and presenting scientific facts, we immediately switch to a different object area, that is, from one special science to another. We have no right to confuse the notions of different sciences, however similar their statements may sound; such a confusion would mean a banal logical fallacy called term substitution: the same word (symbol) refers to different notions within the same discourse.

In fact, this is the essence of the genial Gödel’s idea, to encode all the statements (identified with their formulations) with integer numbers, along with the rules of derivation, so that any statement at all gets reduced to a statement about numbers. But numbers are always numbers, even in Africa; we can compare them in any way, including the diagonal technique. The nature of this logic could be illustrated by an example: let all the even numbers refer to the statements of edible things, and all odd numbers refer to something non-edible; then we can shop for two non-edible things and have a good breakfast just putting them together…

The fraud is even more evident in the following formulation: since all the words are composed of the same characters, they all mean the same.

While mathematics keeps with science, that won’t pose any problem. Every schoolchild knows: each function has a particular domain. For an inadvertent deviation from the appropriate domain, one gets a bad mark at exam. Still, as soon as a mathematician graduates from the high school and proceeds with a career of their own, one can forget about such minor nuisances. Thus, in category theory, many theorems can be boldly proven in the assumption that all the arrows are properly defined in some sense. In the end, we get a result serving to explicitly specify the properties of the arrows presumed from the very beginning. Here is yet another popular fallacy, logical circularity, the other side of the diagonal principle.

A scientific theory is to construct meaningful statements about some application area and establish their adequacy within that very area, that is, their truth. One could formally picture it as a truth function defined on the universe of all the acceptable statements *X*:

$$t : X \to \{\mathbf{F}, \mathbf{T}\},$$

where **F** and **T** are the available truth values (in some cases we need more choices). In real life, this representation is not always valid; but let us accept it for a while. Under the same conditions, deducibility formally defines yet another function:

$$d : X \to \{\mathbf{F}, \mathbf{T}\},$$

admitting that

$$d(X) = \mathbf{T} \Rightarrow t(X) = \mathbf{T}.$$

Omitting the detailed specifications, the fundamental Gödelian statement can be expressed as

$$d(X) = \neg t(X).$$

But the last two functional formulas do *not* belong to the domain of the functions *t* and *d*; their domain is the space of all functions over the universe *X*, which needs an entirely different science… And possibly, it is science at all, if we fail to indicate the object area it pretends to describe.

The mathematical trick of the Gödel theorem, the amalgamation of incompatible statements within the same theory, is widely used in other mathematical sciences. It is enough to recall the great theorem about the stationary point considering the mappings of an *n*-dimensional ball onto itself

$$f : B^n \to B^n$$

and stating that one can always find a point *x*, for which $f(x) = x$. The trick is to blend the domain and the range of a function, though, to remain within science, they can hardly ever coincide, since (using the physical terminology) they are measured in different units: $[x] \neq [f]$. Roughly speaking, they stand on the opposite ends of an arrow, and this positional difference makes them qualitatively different. The very admission of some portion of space transformed into itself is already a petty cheat, since two *different aspects* (or applications) of the same are thus identified. In any case, such a *reduction* of the range of a function to its domain is, in general, a nontrivial operation never imaginable inside mathematics; science borrows such acts *from praxis*. If, in everyday life, we can convert the products of our activity into the raw material for the same activity, you may justifiably use your formalism. If such a reproduction is limited, just be careful with your conclusions. That is why we always need to experimentally test our theoretical predictions which remain mere hypotheses (however necessarily implied by everything we already know) until they are practically proven.

The identification of the range of a function with its domain is akin to the already mentioned fallacy of term confusion. Indeed, even if it is possible to represent the values of a function with the same entities as its arguments, this apparently identical *notation* refers to different *things*. You may wish to mark your fingers with the decimal digits 0 to 9, and do the same for your toes; still, your fingers won't become toes, or the other way round. It is not an easy deal, to make fingers and toes really interchangeable (though, in some particular cases, one can achieve that).

Mathematicians use the word "isomorphism" to justify the tricks like that. It is enough to pronounce the magic incantation that the range of the function coincides with its domain "up to an isomorphism", to drop any further concerns. But, in the real world, to use the output of a system as its input, you need a quite material feedback circuitry; without this practical feedback, reflexive mathematics is of no use. Luckily, the development of mathematics never follows the whimsies of its grands, better listening to the public needs and practical feasibility. Indeed, any time something enters the head of a mathematician, it certainly does not emerge from nothing, and there is a cultural reality that has induced it. However, this reality may be of a peculiar kind, not necessarily in a positive sense. Thus, a mathematical idea may be an expression of a common (methodo)logical fallacy.

For yet another illustration, let's recall computers. Many programming languages distinguish *values* from their *types*. For instance, the number 1 in integer arithmetic is not exactly the same as real or complex number 1; they may be *denoted* by the same character in the language (using the dynamic typing mechanism), but their inner representation will be different and have nothing to do with the corresponding ASCII code, or the string "1". In object-oriented programming, types are represented with classes, while values refer to the instances of the class. A class may have any number of instances, which remain separate objects independent of each other. Still, all such objects are "isomorphic" to each other since their inner structures (fields and properties) coincide, and they allow the same operations (methods of the class). Just try to confuse two objects of the same type in a computer program! You are sure to have a heavy debugging session, and any possible headache. Mathematical bugs are more difficult to trace, though they are basically of the same origin. There is also a psychological effect: the wide public tends to yield to the demonstrations of mathematical "rigor", up to the utter incapability of doubt in the face of some perfect "proofs". This is especially so considering the fact that few people are educated enough to understand mathematical discourse, which is packed up with the references to somebody else's results, to be accepted as a matter of belief.

Well, let us get back to science. I stress once again that any science is to produce statements about some application area, meaning that such statements could be further examined to establish their validity or truth (which is not the same). Traditional mathematics would arrogantly declare all such statements as firmly established, once and forever. That is, any science is only meant to "discover" all kinds of truths. Real life is a little bit different. The development of a science is a complex and dramatic process; there are different directions of research, and one can never predict the outcome of a particular choice. The application area of a science does not exist on itself, like Plato's ideas; it grows along with the growth of the science. To illustrate it with something simple enough, just within the grasp of a mathematician, consider that the *statements* of a theory are yet to be built out of its elementary *notions*. These notions may be either adequate or not so perfect. Similarly, one can construct statements of notions in an either correct or incorrect manner; that is, there are things that can and that cannot be asserted in this particular science. One does not need a microscope to observe that this is yet another instance of the same Gödelian scheme: some meaningful statements are bound to be inexpressible in the terms of the theory, or contradictory. To raise an incomplete and contradictory logic upon such an incomplete and contradictory basis, isn't it just a little bit strange?

In this context, the "linguistic" trick is no longer entirely convincing. Why do they think that the alphabet is fixed for all times? Things like that rarely happen in real life, where we need to invent more and more signs for anything that just did not exist before. In particular, such sign creation can merely *reinterpret* the already existing signs, using different interpretations in different contexts. The effective number of characters will thus grow to infinity, while we are still capable of *encoding* these characters with a finite alphabet. A finite code can refer to (or represent) something infinite, depending on the context, while the number of contexts is in no way limited. With all that, Gödel's arithmetization of logic becomes utterly unfeasible. So, one has to honestly admit that mathematics is a science like any other, with mathematical theories valid only within the area of their applicability, limited to the objects of a certain kind. No more royal pretense; like physics can hardly pretend to be the theory of everything.

Right there, in a truly scientific research, the diagonal principle happens to be quite useful, provided we do no exaggerate the generality of our conclusions. And we can, in certain practical cases, identify isomorphic spaces. Or build, within a limited range, recursive theories and programs. However, if the ends do not meet, and the result seems to be far from the common views, one should not just take the posture of a prophet and blame the dullness of the folks; maybe it is something in the mathematical reign that went wrong. In the latter case, we'll need to drop out some apparently evident identifications and carefully place mathematical constructs in the appropriate levels of hierarchy. Just to introduce some space for mathematics to expand.

Aug 2009

The Straights of the Circle

Unless a person grows into a mathematician, everybody intuitively feels the presence of two opposite sides in any activity: we either proceed stage by stage from one goal to another or repeatedly reproduce the same something, intentionally abandoning the balance to regain it in a little while. This could metaphorically be pictured as the contrast of the straight line and the circle, of translation and rotation. Of course, any reproduction implies production (since one needs at least to do something, to redo it), as well as production is impossible without reproducing the productive environment, the basics of technology (including the typical operations). However, at any instance, identity builds up at a higher level, compared to thus lifted distinction; there are numerous transitions of one thing into another that allow us to establish the unity of the different as a kind of generalization (cultural assimilation). When mathematics sets to ignoring such (qualitative) distinctions in favor of sheer quantitative analysis, logical fallacies are bound to creep in.

Just a simple example. From high school maths, we learn that any periodic (say, with the period 2π) function can be expanded into a Fourier series:

$$f(x) = \frac{c}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right),$$

with the coefficients evaluated as

$$c = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

The constant *c* is commonly considered as a special case of the coefficients a_k, stressing the fact that the expression to compute $c = a_0$ can be obtained from the equations for a_k just formally setting $k = 0$. Everything looks hard and fast, at first glance. Still, there is a minute nuisance: why should we supply the constant term of expansion with a factor of $1/2$? To keep uniformity, we should rather start with a uniform expansion like

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right),$$

taking into account that $\sin(0 \cdot x) = 0$, and hence the value of b_0 is arbitrary (which, by the way, is annoyingly suspicious too)? Yes, we have to introduce an “extra” quotient in the integral for a_0; but why in heaven should we aim at the similarity of formulas for series coefficients instead of original series uniformity? Students are not allowed to ponder much upon such questions; at exams, they must report what they have been taught. Working mathematicians would shyly sweep the doubts under the carpet, with a vague reference to the centuries-old tradition. Still, we perfectly know that new prospects of research were always found where people tried to grasp the meaning of some strange arbitrariness: apparently, one can do either that way or another; later on, one becomes aware of the principles of choice and can consciously act according to practical needs. For a well-known example, take the story of the Euclid’s postulate of parallels. Or Archimedes’ findings. The stray one-half factor in the theory of Fourier expansions may well happen to be of the same crew.

In the preliminary approximation, the constant term of the series is very like an integration constant. It is well known that an indefinite integral is a family of antiderivatives, with the choice of any particular member depending on the additional (initial, boundary, asymptotic…) conditions which do not follow from the properties of the integrand.[2] Well, here is the truism about the integral as a kind of sum, with Fourier series becoming Fourier integrals *etc*. Let’s be less primitive and never agree to the candy box without the candies. It’s high time to dig a little bit deeper.

One could try to ponder on the qualitative distinction of variables from constants: the former are meant to change, that is, to depend on something. A constant does not depend on anything; at most, it may vary along the lines irrelevant for the issue in question. When we hear the common objection that a constant can be considered as a special case of variable, namely, a variable that always takes the same value, this is no serious discussion but rather resembles the old pun of the former soviet dissidents: the Party’s line is straight; it bends at any point… In fact, all mathematical trickery with identifying qualitatively different things exploits a few banal logical fallacies known from the most ancient history.

[2] One could observe a close kinship of mathematical analysis with quantum mechanics; but a discussion of that large topic would be inappropriate in the present narrow context.

For instance, as soon as we admit that a constant is just a constant-valued function, we still have to explain what we mean by "constant-valued". That is, to define a constant as a function, we need to have an idea on constancy (identity) as such. This is a typical case of logical circularity. In that way, one does not eliminate the opposition of constancy and variability, but rather shoves it aside, hanging it up to somebody else's hook. Just we do attaching the one-half factor either to the constant term of the Fourier series or to its integral expression. In other words, a constant is something that remains the same all the time, and it is not like a periodic function that does not only regularly reproduce the same value, but also is meant to regularly deviate from that value; otherwise, it has no opportunity to come back.

To summarize, the introduction of a constant term in trigonometric series contradicts elementary logic and violates the "natural" structure of the series (as a span over an orthogonal basis). As there is no need in such a "zero" term, its particular form is out of the question. Of course, one is free to evaluate the difference of one function from another; sometimes, this difference may be reduced to a mere constant. However, nothing hinders adding to the sum of a trigonometric series any function at all, provided it takes equal values at the ends of the segment [–π, π], or even diverges there, but "in the same manner". Such a background dependence can play the role of the "zero" term no worse than mere constant; moreover, in certain cases, splitting a function into two (not necessarily orthogonal) dependencies may be preferable from the "physical" viewpoint, stressing the objective structure of motion, its hierarchical nature. Obviously, we do not need any uniformity of the background function with the formulas for the Fourier series coefficients; they refer to different entities.

In general, we can split any function (not necessarily periodic) into a sum of the "periodic" and "non-periodic" components: the former is obtained as a sum of a Fourier series; the latter plays the role of reference level for any oscillations. Obviously, the "non-periodic" component, in its turn, may exhibit certain oscillatory behavior; however, the characteristic periods of such oscillations will normally be rather large, on the scale of the earlier established periodicity, so that all the "local" oscillations will average into zero during one higher-level period. In this way, we unfold a hierarchy of oscillatory motion, and it is only in very special (practical) cases that it can be reduced to something planar. It is also evident that every such hierarchical structure can be folded and unfolded into a similar structure with different sequence of levels, which, too, is determined by applications rather than purely mathematical necessity. Eventually, we come to considering various ensembles of the possible hierarchical structures as virtually co-existent; this will shift the focus from the details of inner motion onto its global organization, picturing the hierarchy as a whole, a unity of all its special positions.

Returning to the metaphor of the circle and the straight line, we find that, at each point of the circle, the motion has a definite direction represented by a straight line; still, any physicist knows that such ("virtual") displacements lie in a different (tangent) space, and we cannot arbitrarily identify the points of the configuration space with momenta. Similarly, one can "straighten" a circle representing it with a sequence of small segments of a straight line. However, such a picture is only acceptable where the difference is practically irrelevant; formally, the indication of the adopted scale (the level of consideration) is referred to as convergence to a limit. That way or another, the removal (lifting up) of the opposition is a quite real operation; once we forget it, we get stuck in logical problems.

By the way, a few words about symmetry. To be honest, $\sin(0 \cdot x)$ is nothing like identical zero, so that omitting the terms with b_0 in the Fourier expansion is only possible in a limited area. Proceeding to infinity, we come to an indeterminable form of the type $\sin(0 \times \infty)$, and it is not at all evident that it will always evaluate to zero. Students are often told that a trigonometric series is defined in the interval (–π, π); this is a deliberate fraud, since both the sine and the cosine are defined for any real numbers, and hence the sum of the series will exist everywhere. It's quite another matter that the sum will necessarily be periodic in the absence of "zero" components, which are the only way to explicitly introduce any trends, albeit reducible to mere constant shift. In other words, periodicity is an entirely local phenomenon (since we compare the states of the same object, the time points); in infinite domains, it may become just anything. From everyday practice, any programmer learns that the sine and the cosine are only reliably computable for relatively small arguments; the reduction of very big numbers to a standard interval leads to a significant loss of accuracy.

There is a much more fundamental reason to omit the "zero" terms in Fourier series. To the matter of fact, zero is not entirely a number. Indeed, it is not a number at all. It is merely a common notation for the limit process, a placeholder for the (vaguely felt) boundary of the applicability area. To include zero in the set of natural (or real) numbers is, for a person of reason, an act of violence against logic.

In any mathematical (or any other) theory, zero denotes the absence of the object (or an effect), that is, passing beyond the object area. Formal extrapolation of a theory beyond its applicability region is not always justifiable. Infinity says about practically the same: here, we take the object area as a whole as a specific (higher-level) object that is different from any object belonging to the theory's domain. Conversely, zeros refer to the lower levels of hierarchy taken as a whole. Due to hierarchical conversion, the distinction of the "upper" and the "lower" is relative; that is why, in formal theories, zero readily tends to produce infinities, while infinity get inverted into zero.

In a sensible (logically consistent) theory, the natural number sequence starts with unity; this is the first natural number (indicating the presence of something to count al all). Zero is not a natural number, as it is just a contracted notation for *not belonging* to the set of natural numbers. In the same manner, we can only speak about trigonometric series when we get at least one trigonometric function; adding non-trigonometric terms drives us beyond the theory of trigonometric series to a different theory (if not to an eclectic mixture of theories). Similarly, in power series expansions, the "zero-order" term is nothing but a reference to outer conditions, the other levels of hierarchy; anyway, this is beyond the strictly understood object area. Considering a problem "in the zero order", we, in fact, do not consider it at all, rather preparing the very possibility of consideration by preliminarily outlining the object area; it is after this preliminary work that we can proceed to discussing the object in first order, adjusting the details at higher levels.

At school, we learn about various smart "theorems" concerning the convergence of trigonometric series. For instance, it is said that a continuous function in the segment [–π, π] with no more than final number of extremums has a Fourier expansion that will converge everywhere in this segment, so that the sum will coincide with the function's value in every inner point, while at the both ends of the segment, the series will evaluate to

$$\frac{1}{2}\left[f(-\pi) + f(\pi) \right]$$

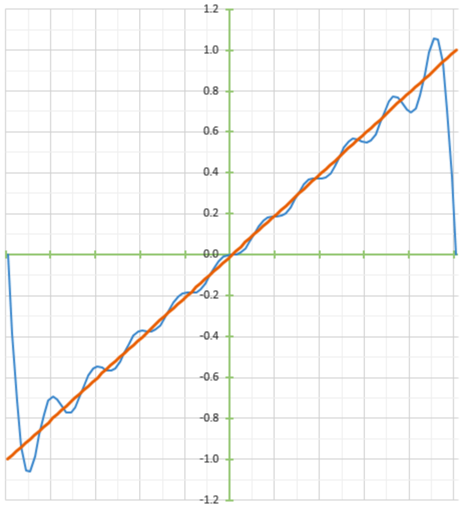
The typical school example is linear dependence $f(x) = x$, with the Fourier transform, for convenience, reduced to the interval (–1, 1), given by

$$f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin \pi k x$$

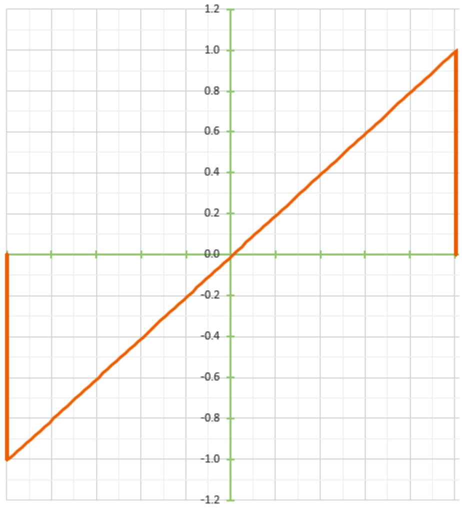
Formally substituting $x = \pm 1$, we obtain (with the epithet "evidently") that every term of the series identically equals to zero; on this reason, we dare to conclude that the whole sum will also compute to zero, thus brilliantly confirming our rule about the ends of the segment… In reality, this derivation contains an elementary logical fallacy; normally, students get punished with poorer marks for such blunders at exam. Indeed, the terms of the series are only equal to zero at $x = \pm 1$ for finite *k*; in the limit, however, we come to the familiar indeterminable form of the type $\sin(0 \times \infty)$, which has to be honestly analyzed and brought to a specific value. The superficial decrease of each term as the inverse of the index is bluntly compensated by the increasing number of terms, so that here, as well, there is no loophole for trickery.

Surely, this is not an incidental lapse of reason; in fact, we speak about a most profound logical problem. Conventionally, one might call it the substitution principle. Mathematicians suppose all the way that one can (at least in principle) to substitute any term in any formula with anything else, and nothing will change. As we observe, this does not always hold. In general, one cannot evaluate an infinite sum substituting some specific value for the parameters (arguments) in every single term of the expansion. The trick will work under certain conditions. However, in real life, the result of a substitution often depends on its practical realization (for instance, on the order of substitution and the technique of producing the partial sums). For recursively-defined formulas, momentary substitution becomes utterly meaningless; any operation must refer to a specific stage of recursion. Well, the bulk of problems arising from the substitution principle deserves a separate discourse.

Now, what are the limit values of a Fourier series at the ends of the one period-long segment? A catchy question. Yes, one could simple abandon the idea of computing the sum at the ends and believe that the limit at $x \to \pm 1$ does not exist, so that only the inner points in (–1, 1) are legal. Just to lull one's conscience, the single-ended limits could be considered. However, this is no bright perspective, is it? Let us, instead, look closer to the graphs of the partial sums of the trigonometric series:



Here, it is literally evident that every partial sum is representable with a smooth curve, and there is no reason to expect any discontinuity in the infinite limit. Anybody alien to mathematics will say that the graph will "topologically" converge to the polygonal line:



Yes, this function is not single-valued, and the arguments $x = \pm 1$ map to a continuum of ordinate values. Still, who can prevent us from slightly rotating the axes, to arrive at perfect unambiguity? Geometric forms do not depend on the method of their arithmetization. The curve (trajectory) remains the same geometrical entity however parametrized. It is the forms of representation that change, but not the represented objects. The converse holds as well: the same numerical structures can correspond to very different objects (which may be very far from mathematics in real life). Any representation is conventional, partial, and valid within a particular approximation (in a specific context). Still, at least some representation is always possible and practically inevitable.

By the way, the trick with the displacement of the coordinate system axes is not new: thus, in complex analysis, we have long since grown accustomed to such manipulations in the vicinity of pole singularities. So, should we be any shyer with Fourier expansions?

It is important that the Fourier series for any "decent" function $f(x)$ will converge to some continuous (and, of course, periodic) function $F(x)$, which, in general, does not need to coincide with $f(x)$. To eliminate the formal "leaps" at the ends, one could resort to a parametric form of the incident function (say, with the path length as a parameter):

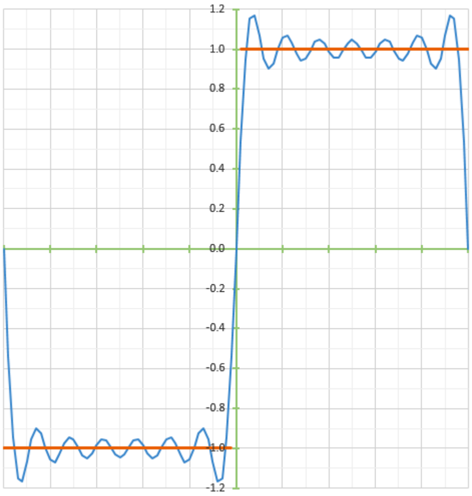
$$x = x(s)$$
$$y = y(s)$$

In this representation, there is no longer any discontinuity, and the apparent "leaps" get "filled" in a natural way, without special effort. However, one still cannot get rid of the problem: with more terms in the expansion, the sum will rapidly oscillate (albeit with a quickly decreasing amplitude), so that evaluating the limit is not quite trivial; instead of the indefinite terms of the $\infty \times 0$ type at the ends of the segment, we get the same indeterminacy in every inner point! [3] To separate the trend from variations, there is a standard trick: we "smoothen" the curve, eliminating too fast oscillations:

$$F_N(x) \to \tilde{F}_N(x) = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} w(s-x) F_N(s) ds$$

The oscillations with the periods much smaller than Δx will gracefully average to zero, thus producing a "good", smooth curve. With more terms in the Fourier expansion, the averaged function will converge to the same limit $F(x)$, without annoying indeterminacies. Still, in some cases, it is the deviations from the average that are of primary interest. The theory of scaling in music provides a practically important example.[4]

The Dirichlet theorem could be discussed in exactly the same lines. Thus, from the graph for a partial sum of the Fourier expansion for the function $\operatorname{sgn}(x)$



we, once again, perceive convergence to a polygonal line, with the discontinuity of the original function filled in the limit by a vertical segment of the ordinate axis. Instead of a single leap, we get a continuous meander. And this is right, since the sum of periodic functions with commeasurable periods is a periodic function; and we expect it like that, intuitively opposing the straight line and the circle as qualitatively different forms of motion. A graph of a periodic function is a sort of involute of a circle (or many circles) along a chosen line.

[3] Something of the sort has once lead to the birth of quantum mechanics.

[4] L. Adveev and P. Ivanov, "A Mathematical Model of Scale Perception", *Journal of Moscow Physical Society*, v. 3, pp. 331–353 (1993)

Despite the formal convergence of a trigonometric series within a finite interval, we cannot say that it will converge to the incident function. The minute details of the hierarchy (in particular, the hidden behavior of the partial sums) are in no way lost in the result. That is why it is not always reasonable to compute a function using its Fourier expansion. And it is not the matter of slow convergence or apparent singularities. Any science speaks about things outside us (however preconditioned and intertwined with our material activity); so, the structure of science must correspond to the structure of its object. Where the very organization of human activity prescribes the usage of Fourier analysis, we must employ it, abandoning the hope to simplify things with apparent principal trends or global features to suppress less important variations. For example, if a linear function or a step are produced by some electronic devices, the wave nature of the process will practically manifest itself one way or another; here, trigonometric expansions are all right. On the contrary, considering a technological process, an evolution of a star, or a banal movie show, we find a linear approach (though, possibly, accounting for various reflective loops) more natural and self-suggesting. The opposites do not exist one without the other; all we need is to seek for the right place for everything.

Well, the notion of a limit (including the limit of a series or an integral) is not as trivial as it may seem from the school course of mathematical analysis. We have to consider convergence of an object to an object, rather than mere numerical convergence (which, however, may be an important special case). A function can be arithmetized ("computed") in many ways; no single arithmetization will express the idea of the function as such. Our examples of "topological" (or "extensional") convergence are in no way a complete enumeration; they merely stress the difference of the operational definition of a function from its extensional definition (a "graph"). There are many more types of definition (like implicit, schematic, illustrative, applied) that cannot be reduced to neither operations not sets. The hierarchy of all the possible definitions is a function proper.

It is self-understood that any revision of the notion of the limit will introduce certain specifications to the theory of differentiation or integration. No serious shock for the whole of mathematics, of course; still, it may be useful to cast a fresh glance to an old moss-grown domain.

In a specifically practical aspect, the idea of a graphic limit may be of value where the traditional approach states the absence of a limit or suggests an arbitrary closure. It is quite possible, that, in such cases, there is no arbitrariness at all, and the apparent lack of convergence is due to a poor parametrization. One could informally conjecture that all the known "exotic" functions are nothing but examples of inconsistent (unnatural) arithmetization; with a more sensible approach, all the uncommon features will disappear, and everything will get back to the combination of the two simple types of motion: the straight line and the circle, translation and reflection.

Numeral Systems

Despite of the stubborn attempts of the apologists of "pure" science to detach mathematics from life, any mathematical solutions are only meaningful in the context of certain practical needs. Sometimes, it does not go much beyond preliminary exercise, similar to children's play; still, acquiring a sort of "activity mood" is most important to proceed to activity proper. Most mathematical results prove viable inasmuch as their parent forms of activity keep on somewhere in the hierarchy of the culture.

Just for illustration, a few well-known facts from the life of numbers.

We grew so accustomed to these little creatures that we just don't admit a thought about their origin and nature, as if they had existed long before the humanity's arrival to the Earth and would exist after it's gone. Yes, numbers express something objective, the essential in things that does not depend on whether anybody thinks of it or not. However, they are but a form of expression, the subjective view of objective quantities. Numbers come as a product of human activity, so that specific activities may lead to the different ideas of a number.

As in any science, mathematical forms either refer to the observable properties of things, or are produced in the inner development of science (theoretically), or are used to delimit the domains of the possible applications (as a typical experimental setup). In particular, a number may characterized the interrelations of different activities; this type of comparison is yet another activity called *measurement*, as one activity becomes a measure of the other. Measurement cannot produce just anything: on one hand, it is determined by the object, and on the other, we need a practical result rather than the process of measurement as such, which leads to a limited accuracy of any measurement within the range of reasonable sufficiency. That is, we approach the complexity of the world with a subjective scale, using the historically formed reference frames. When a scale is no longer satisfactory, we build yet another one, which does not eliminate any previous valuations to be held within the new estimates as the levels of approximation (or alternative pictures of the world). There is no need in "exact" values where some general judgment is enough. Moreover, certain scales may fail to go along with each other. Thus, Europeans usually distinguish four seasons of the year, while some people in Asia and Africa prefer to speak of five seasons (possibly of variable length). The both views reflect certain aspects of the objective reality; still, it would be rash to judge of European climate by African measures, and the other way round.[5]

A typical measurement procedure is hierarchical: first, we develop a general idea of the scope; in the following, facts get sorted out according to some specific criteria; finally, we may need several degrees of refinement, subdividing the gross partitions into smaller portions. In mathematics, this process is reproduced in a specific activity, and we call its abstract product a system of numeration.

In general, a numeral system is a hierarchy of scales (gauges), so that any number (from a practically important range) could be represented by a collection of positions on the different levels. Conversely, given a collection of such valuations, we can organize our activity and tune the measurement system to produce exactly that numeric result.

There are no restrictions on the nature and structure of the possible representations. Each of them fits in an appropriate practical area. Similarly, the same number admits multiple representation, as it can be obtained in the course of very different activities. The issue of the equivalence of such different representations thus stops being a favorite mathematicians' toy and become a quite pragmatic question: where several activities are involved in an embracing activity, their results are objectively commeasurable. No such unifying activity, no sense in formally introducing any relations. In particular, formal constructions may service the activity of consciously rearranging production technologies; in this case, mathematical results will acquire a normative air: we do not merely adapt our behavior to reality, but rather dare to demand that reality agreed with our expectations, and we actively interfere with the affairs of the world to make it keep our environment within the suggested parameters.

For the moment, of all the common numeral systems, positional systems might be considered as the most universal, with the rest kept for mainly historical reasons. Time will show whether this viewpoint is viable. Currently, there is a certain trend towards more flexible positional representations; nobody warrants that no further generalizations shall come. In the traditional setup, a positional base K numeral system is associated with the ground scale constituted of the natural numbers $\{1, \ldots, K-1\}$, so that a (positive) result of some measurement could be represented by one of the numbers from this finite set (namely, the "closest" in the given context, in the sense of the current activity). If the quantity does not fall in the ground scale, we proceed to the next level of hierarchy, consecutively introducing scales of the type $\{K^p, \ldots, (K-1)\cdot K^p\}$, with any natural p, for numbers greater than $K-1$, and the scales of the model $\{K^{-q}, \ldots, (K-1)\cdot K^{-q}\}$, with natural q, for numbers below unity. Once an appropriate scale is found, the closest element of the scale is meant to be the *first significant digit*, while the corresponding number p or $-q$ is referred to as the *order of magnitude* (or a *position* of the record; hence the term "positional system"). In many cases, knowledge of the orders of magnitude is enough for decision-making. Otherwise, we examine the difference of measured value from the first rough estimate. Provided the procedure of measurement is flexible enough, we can estimate this difference by the order of magnitude thus obtaining the next significant digit. In this way, any (measurable) number gets eventually represented by a collection of pairs $\{k, p\}$, or $\{k, -q\}$, where k is the first significant digit in the corresponding order. In general, the result of a measurement does not need to be represented

[5] Of course, global climatic shifts may induce a revision of our notions of the seasons, likewise in Europe and in Africa.

in all the orders: each number unfolds its own hierarchical structure with a specific set of nonempty positions. Conventionally, it could be written as

$$k+\sum_{\nu=1}^{N} k_{\nu}^{+} K^{p_{\nu}} + \sum_{\lambda=1}^{\Lambda} k_{\lambda}^{-} K^{-q_{\lambda}} \ , \tag{$*$}$$

where each of the three parts may be missing. For convenience, the non-significant elements are commonly denoted by a special character, zero. One is to be clearly aware that zero is *not* a number; it only marks a vacant (void) position, just like in a payment order form, where the field for the total to pay is marked out by a row of empty cells to fill in. Nowadays, mathematicians tend to include zero in all kinds of numerical sets, with a lot of surprise at the freaks of the theory due to this logical laxity. Still, who would not be seduced? With a formal inclusion of the zero in the collection of admissible digits, we can rewrite ($*$) in a simple and attractive form:

$$\sum_{\mu=M_{\min}}^{M_{\max}} k_{\mu} K^{\mu} \ , \tag{$**$}$$

with the summation running over all $M_{\min} \le m \le M_{\max}$, admitting that some positions are zeroed out. A convenient notation is a great thing; with appropriate conventions, some implications become just obvious. Nevertheless, any notation is designed for a certain class of activities, and it will limit the thought to that very class, up to obstructing scientific creativity with a pile-up of stereotypes.

Clearly, a positional system is best suited to representing rational numbers: any measured value can only be written as a finite sequence of digits (with a finite "accuracy"). When a mathematician urges us to fancy an infinite sequence, the magical words "and so on" are but a sheer emptiness unless we can indicate a specific activity to produce new members of that infinite row. In fact, infinity is beyond mathematics; this is a reference to an unachieved activity, the absence of the product. In the sense of our template scale, zero and infinity are the two complementary kinds of its inapplicability, for too small or too big numbers (meaning that, for practical reasons, we choose to restrict ourselves to linearly ordered scales).

Strictly speaking, the quantities of different levels are not comparable, and one has no right to bluntly sum them up. The sums in ($*$) and ($**$) are but a technical convention, a way to stress the hierarchical nature of activity and the corresponding scale. To literally understand such formulas is like add a bike to an orbital station: yes, Russian rubles can sometimes be converted to US dollars, but this requires yet another activity, with an in-built overhead. As our mathematics does not much care for formal discretion, one can use this super-democratic science to evaluate the level of public wealth by taking the average of the meager pennies of a pauper and a round capital of a multibillionaire, which is busily utilized by the bourgeois brain-washers.

With all that, we shall follow the tradition for a while, permitting hierarchies squeezed to a flat something, so that any numbers, regardless of their source, could be considered as comparable. Sensible abstractions make no harm. Things are getting rude where a sketch of a landscape is meant to be the landscape itself, and reasonable choice is to cede to despotic arbitrariness.

The notion of a limit is the next step towards more formality. We cannot sum up the whole infinity, to write down all the digits, accounting for every finer (or coarser) scale. Still, in many practically important cases, one can find that different hierarchical structures represent the same real object; in the context of positional systems, we speak of the levels of accuracy, holding in the mind the possibility of less trivial interpretations.

Naturally, any unity means the presence of a unifying activity. Traditionally, mathematicians compare two numbers written down in the same positional system and say that these numbers differ by less than K^M if their records coincide for all $\mu \ge M$. For negative M, this refers to the segments of the fractional part, so that the "exact" value could be defined as the limit at $M \to -\infty$. As one will readily guess, the segments of the integer part can be treated in exactly the same manner, and numbers can be compared by coinciding strings for $\mu \le M$; infinite growth is in no way different from infinitely approaching zero.[6] In hierarchical terms, we say that any sequences of digits coinciding between the levels $M_{\min}$ and $M_{\max}$ represent the same number *in these limits*. This does not prevent us from discovering differences in other limits; any equivalence is therefore relative, and a consistent theory must explicitly indicate the hierarchical structure supporting that particular way of identification. In modern mathematics, however, the domain of applicability is only implied, which stirs the temptation to declare a special result as a universal law, the ultimate truth valid for everybody for all times.

Practically, the existence of a limit means feasibility, that is, we can produce a particular product in a properly organized activity. Quantitative comparison of such products is a separate activity; if, for some reasons, one quantity cannot be expressed through another, this is not a mere formal irrationality, but rather an expression of certain objective arrangement of things. Practically comparable objects (the products of the same activity) must be mathematically comparable, provided an appropriate number system has been chosen. In other words, such things belong to the same scale hierarchy. Thus, the fraction 1/3 contains an infinite number of digits in the decimal notation; in the positional base 3 system, it has a finite representation, while the fraction 1/10 becomes an infinite sequence of digits. The commensurability of these quantities comes out when we consider the positional base 30 system, where the both fractions are finite. If the base 30 hierarchy happens to be unachievable (forbidden by some "selection rules"), no mathematics will allow comparison of ternary and decimal numbers in practice. Situations like that are well known to musicians, as the different musical pitch systems (scales, moods, tonalities, chords *etc*.) are not always compatible within the same composition.[7]

In this context, the incommensurability of rational and irrational numbers only means that some numbers (characterizing the products of certain activities) cannot have a finite representation in a positional system with any *natural* base K. So, what? In the XX century, many alternative positional systems have been thoroughly studied, in particular, with a real base and real numbers for the "digits".[8] The hierarchy of scales is in no way restricted to the clones of the same scale; in general, this is collection of "natural" (for a given object area) measurement units (which may differ at different levels), with the corresponding sets of compatible scales (selection rules). In physics, by the way, we normally adopt the unit systems that eliminate most dimensional factors in the formulas. The same policy is tenable in mathematics as well, as we demand that a consistent structure of a numeral hierarchy (a positional system) should reflect the hierarchy of the corresponding object area (the organization of activity). If a product of activity is practically feasible (obtainable in a finite time), its mathematical models will be finite as well, with an appropriate choice of the scale hierarchy. No limit convergence is needed any longer.

Mathematical incommensurability is an expression of the objective qualitative distinction. For instance, take a cylindrical volume (a "glass") with the base diameter 1 and the height 4; its capacity equals π. Now, take a rectangular volume (a "box") sized as $1\times1\times3$, with the capacity of 3. Using one of the volumes, one can never produce the same quantity of liquid as using another. But, who can forbid us using both volumes in alteration, depending on the production necessity? The total quantity of liquid will then be expressible as a pair of numbers rather than reduced to a common scale: the number of glasses + the number of boxes. Here, the plus sign corresponds to a quite real operation of mixing the content in a volume of a high enough capacity. This closely resembles what we do in quantum mechanics, where a state vector of a two-level system is represented by a linear combination of the "pure" states, with the observables including the contributions of the both components. Reduction of many-dimensioned scales to a common base is only possible with some finite accuracy acceptable for a

[6] Formally, one could associate any integer with a regular fraction replacing the positive powers of the base by the corresponding negative powers. A general number is then represented by a pair of fractions, and one can easily define a kind of "non-standard" arithmetic on the set of such pair to preserve the common properties of real numbers. Then two numbers are called K-dual if the "integer" component of one corresponds to the fractional component of the other, and *vice versa*. For instance, in decimal notation, the numbers 3.14 and 41.3 are dual. K-duality is a special case of hierarchical *conversion*; such operations reflect the different modes of practical activity.

[7] L. V. Avdeev and P. B. Ivanov, *Journal of Moscow Physical Society*, **3**, 331–353 (1993).

[8] *E.g.*, see G. E. Edgar, *Measure, Topology and Fractal Geometry*. — Springer-Verlag N.Y., 1992.

practical purpose. Provided there is a real procedure of generating finer approximation of that type, one can refer to it as approaching a limit. Still, there is no reason to believe that every aspect of the world is expressible in numbers, and even less, that a limit of a number series will necessarily be a number.

For an illustration, consider a positional system with a complex base $Z = K\exp(i\varphi)$, $|K| > 1$. Then Z^{-m} formally tends to zero at $m \to +\infty$, and one might fancy constructing fractions of infinitely increasing accuracy. However, as the phase factor $\exp(-im\varphi)$ is rapidly oscillating for higher m, the zero limit value of the amplitude corresponds to a continuum of the possible phase values, which is far from being a number. Different modes of approaching the limit will result in specific phase distributions.

Above, we only discussed the structural (static) aspects of numeral systems. In addition, one could also touch the system side proper, system dynamics: a general system is exactly the way of producing structures by other structures, so that some input would result in a certain output. Thus, we must be able, given a number, to construct its representation in the current positional system; on the other hand, we need to use a positional record of a number to direct our activity so that its product would be characterized by that very number. To get rid of such routine operations, humans invent all kinds of automata mechanically reproducing what once was a creative inspiration. Of course, the robot does not care for shifts in the world resulting from its operation. To assess the outcome, one needs a human. At least, we need to decide about the appropriate level of activity, which will shape out judgement of whether there is any result at all; it is only afterwards that we can get engaged in comparing the products. This nontrivial task gave birth to the huge edifice of the computation theory (including quantum computing). First, we gingerly examine the cultural heritage of the traditional computational techniques; eventually concocting a general principle, we start to dogmatically impose it all around as an *a priori* criterion of truth…

In the theory of generalized number systems (with real base and real digits), one finds a theorem stating that, for any choice of the base and digits, the representation of the complete real axis will be either ambiguous (some numbers will be represented by at least two sequences of digits), or incomplete (with some real numbers not representable in the chosen numeral system). This closely reminds the situation with the famous Gödel theorem about the logic of completeness and non-contradiction. The resemblance is not accidental; it is to stress once again that mathematics is essentially rooted in human activity, albeit squeezed in the narrow normative patterns, official standards, or public stereotypes.

Now, let us possess an automaton to feed in a (non-formal) number and (with a little patience and luck) catch an eventual outcome representing the number with a sequence of characters (the digits of the generalized numeral system). Since, traditionally, the collection of the digits is finite, one could somehow order and enumerate them; thus we come back to the natural-number representation with the digits from 1 to K–1, except that the number K is no longer taken for the base. On the next stage, we shall extend the original collection of digits adding any finite sequences of the ground-scale digits (possibly including zeros in some positions) as separate digits. Such fragments of a positional record are countably denumerable, and we can label them with natural numbers starting from $K+1$. These new digits are no worse than the original ones and we can consider numeration systems with (countably) infinite sets of digits corresponding to the possible states of our numeration automaton. This may be more technically handy, to avoid any commensurability issues; we never need to seek for common scales, since any rational number at all is representable in thus extended system with a finite sequence of digits. In the systemic picture, the machine calculates the positional representation of a number digit by digit and stops after a finite number of steps. The last output digit will be interpreted as infinitely repeated, since it will stay the same in all the subsequent checks (regular time moments). In particular, that last digit may happen to be zero; the traditional finite expansions are thus reproduced.

In this way, the issue of irrationality gets reduced to the typical question about whether the automaton, once put in motion, will stop at some step or not. A minimal acquaintance with computation theories hints that the problem does not have a formal solution. Instead, we get a hierarchy of complexity levels, with poorly understood (and sometimes poorly defined) interrelations. The worst of all, the very notion of a machine at rest can hardly ever be formally defined. Indeed, let the automaton stop operation and display the same output ever since. How do we know that it won't resume activity after a very long time? There are expansions with extremely long sequences of repeated characters. Can we claim to obtain the final result, or shall we wait? How long? A minute, a year, an eternity? The answer can only be practical: once we feel that something remains unchanged during a particular activity, we may consider it a constant, an axiom, or a rule (a law). Such an approach is common among physicists, speaking of those who did not yet go crazy about "accuracy" and "rigor" and make mathematical "truths" a religious belief. A factor in the integrand that is almost constant in the integration region can be boldly taken out, possibly replaced with an appropriate average. All we need is to decide something for the time being and proceed with the activity, rather than sit still with a vacant gaze upon the stupid machine. If our decision goes wrong, life will correct us where needed. Making errors is better than no making at all.

Our preliminary and scarce ideas of numeral systems are also bound to evolve in most unexpected directions. For instance, the uniformity of the scale at any level is highly questionable, and hence the notion of the base needs a thorough revision. We use a "mixed" scale every time we consult a watch. It seems quite expectable that, to measure something with on the different levels of accuracy, we will need as different technologies producing their native measures. Following the reflection rule, we'll need as diversified mathematical models.

Now, every hierarchy is qualitatively infinite, and one can always consider intermediate levels between any previously found. A numeration system does not need to be unfolded in a discrete hierarchical structure, in a sequential manner. That is, instead of the sum in (∗∗), one might employ an analogue of the Lebesgue integral:

$$\sum k_\mu K^\mu \to \int k(\mu) K(\mu) d\mu \to \int K(\mu) dk(\mu)\,,$$

with every distinction level $K(\mu)$ (a "position" in the current notation) introduced through a measure *dk* playing the role of a digit in that particular position.

Yet another generalization will drop power series expansions in favor of some generalized expansion base. For instance, one might try any set of orthogonal functions, like in a common Fourier series. Many exotic constructions go in the same row: exponential and factorial expansions, the binomial system, Fibonacci numbers *etc*. It is not evident whether some of these systems are practically justifiable, but they may well serve as yet another mathematical game (why not?). Just for fun, observe the strange reincarnations of the same positional record (a sequence of digits) in very different numeration hierarchies. In certain cases, such an amusement might grow beyond the game. Thus, a "spectral" representation of a number implies a definite sequence of "energy levels" summing up into a kind of "inner energy".

Finally, the persistent attention to the cultural roots of the numeral systems, their derivation form the human activity, will inevitably bring us to the idea of culturally determined scale hierarchies eliminating any arbitrariness. Once again, one might recall pitch system formation in music. The historical selection leaves us with stable and regular enough scales, suppressing any virtual fluctuations. Each scale is a discrete collection of continuous zones, implying a well-determined hierarchy of subscales, embedding some zone structures in the other scales in a logical manner. The possible ("permitted") musical forms depend on the structure of the fundamental scale, so that music written in a different mood will sound differently. This model illustrates the significant features of any activity; hence, the development of mathematics will gradually reveal such natural structures. Of course, the traditional mathematical constructs will stay; they are obviously useful in many applications. Still a sensible science would avoid both empty zeros and wild infinity.

Perceptive Forms

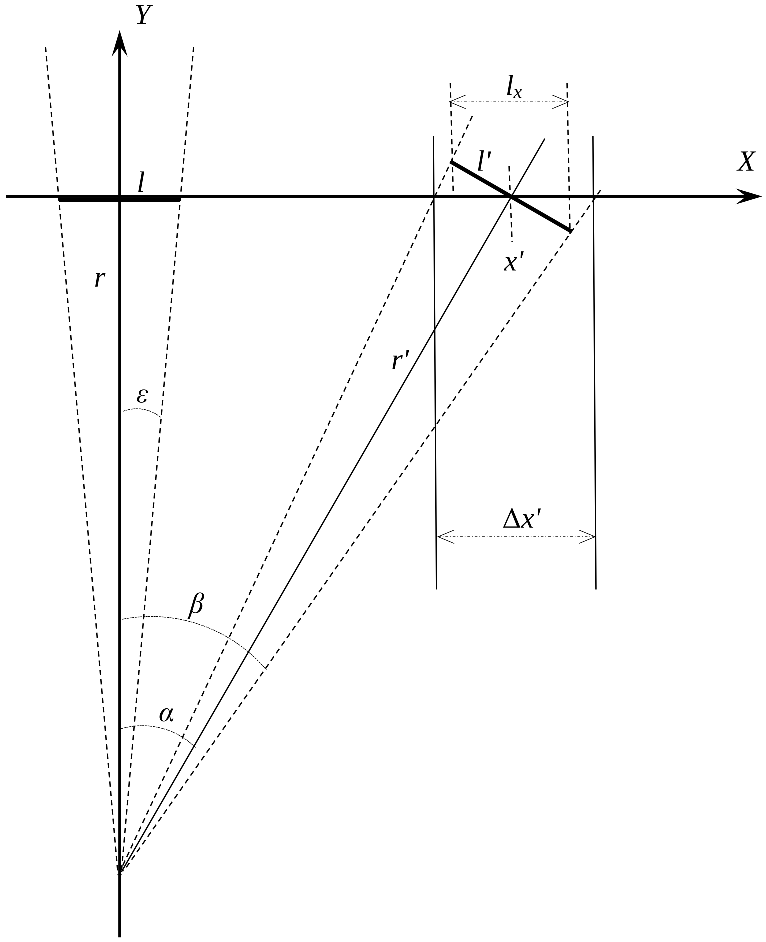
The abstract mathematical space indifferent to anything that might happen inside it is one of the greatest achievements of the human mind demonstrating our ability to break free from the barriers of our inherent finality and aim at comprehending of the world as a whole, with the side effect of clarifying the humanity's place in all that immensity. Still, as long as we need to take care of our local affairs, there is no disgrace in accounting for what and where we are, to sensibly weigh our real capacities and never pretend to anything beyond the reasonable; otherwise, what's the use of our reason? Within this limited universe, one surprisingly finds that bare mathematical space is not always the best idea we might have about the reality around us, and that the formal simplicity has to be augmented to an ugly *ad hoc* construction, to be practically tractable and feasible. Such a tailor art may gradually transform into a common habit and everyday routine, with this abstract formality being quite fit for becoming yet another mathematical theory, which, in its turn, cannot be satisfactory in all respects and demands an informal extension. That's right, life never stays the same, and our science has to keep pace with cultural development.

The principal difference of the perceptible world from an abstract mathematical concept is in that nobody ever gives us a ready-made space in all its entirety: we have to repeatedly reconstruct our space adjusting it to the current activity. Of course, nobody starts from scratch: there are numerous templates and half-products, so that one can always find something of the kind and combine a preliminary picture of the new spatiality from the already known building blocks. This original metaphorical models will later accumulate a lot of corrections and specifications, to become a powerful operating tools; in a little while, people will manage to derive all that eclectic appearance from a general idea, thus adding yet another universal technology to the stock.

For an illustration, let us slightly spice up our vision of the trivial everyday space. Since the beginning of time, we treat it as three-dimensional. The three dimensions are basically equivalent for mathematics, while humans not only find them qualitatively distinct, but also stress the difference with a choice of names, varying from typical situation to another: "length—width—height", "length—width—depth", "width—height—depth", "distance—azimuth—declination", and so on. When it comes to placing one thing inside another, two parallel parameter kits are considered, with the appropriate selection of the names. Human perception is never passive, it represents the outer world in terms of human activity. To look at a thing to the left or to the right of us, we need to turn to the corresponding direction by an appropriate angle; to evaluate height, we need at least to tilt the body, or maybe go up/down; to measure distances, we need a quite different instrumentation, selecting between varieties each suited to a particular spatial range; as for the depth, we often cannot immediately grasp it at all, which leads to a wide usage of indirect methods (the same holds for the objects beyond the perceptive horizon).

A formal model of the perceptive space will necessarily differ from a model of the space "as it is". Admit that some finite-sized thing is placed at some distance r from the observer. This phrase already involves a very strong abstraction: since, in general, the distinct parts of the thing imply the distances of their own, we need to both take the thing as a whole (which refers to some average, or integral, distance from the observer) and consider its spatial organization, with a number of well-discernible details that can be inspected one by one (at least, to produce an impression of integrity, to present the object to our senses so that we could recognize it as that particular thing). The activity of observation will therefore unfold itself into a series of observation acts, each presenting us a part of the whole, namely, the one appearing in the currently formed "focus of attention". This zone of immediate grasp cannot, of course, be reduced to a single point; sometimes it covers a spatial zone vast enough to absorb the whole object, which, in this case, is perceived as a "point"; still, most often, one has to move the focus of attention within some global "field of vision" in the attempt to find the edges of the object (or some other border features). As a result, the object gets represented by the structure of the process of examination. In real life, one rarely sticks to a single file of fixation points: the total time of observation normally allows tracing many virtual paths and thus, to certain extent, get free of the superficial impressions to come to a relatively stable and more consistent vision of the thing.

To make it clearer, let the observer's look stroll in just two dimensions; a segment of a straight line with the center at the distance r from the eyes will do as an object to observe; let it lie in the horizontal plane and be orthogonal to the (shortest) line of sight. Then any single act of observation means slightly turning the head to the left or to the right within the same plane. In every such act, at each angle of sight, the focus of attention covers the points of the object segment that do not much deviate from the line of sight (let this spread be characterized by a small angle ε). For the observer, all such points form a segment of line with the length l; for the direct look, this coincides with the range of point within the segment to observe: $\Delta x = l$. Always assuming the validity of the laws of the linear optics, one could draw the picture of observation as follows:



Certainly, to get a non-zero angle of sight one has to account for both the optical form of the eye and binocular vision; the latter is also important in estimating the distance to the object. However, such physiological considerations have nothing to do with perception as such, since it is a kind of activity, that is, a phenomenon of a higher level, compared to any biology. The bodily organization of perception may largely vary, up to elimination of any organic components at all.

When the line of sight gets tilted by the angle α the focus of attention moves to the point x', thus presenting a new fragment $\Delta x'$ of the object to the observer, who perceives it as a segment with the length l' orthogonal to the line of sight, at a somewhat greater distance r'. Since we are to keep within the original straight line, the sight cannot be deflected too much, and $\beta = \alpha + \varepsilon < \pi/2$. Obviously, $l = 2r\,\mathrm{tg}\,\varepsilon$, $x' = r\,\mathrm{tg}\,\alpha$, $r' = r/\cos\alpha$, $l' = 2r'\,\mathrm{tg}\,\varepsilon = l/\cos\alpha$. That is, with greater α, we can grasp a longer segment of the object line at once:

$$\Delta x' = r\left[\mathrm{tg}\,\beta - \mathrm{tg}(\beta - 2\varepsilon)\right] = \frac{r\sin 2\varepsilon}{\cos\beta\cos(\beta-2\varepsilon)} = \frac{2r\sin\varepsilon\cos\varepsilon}{\cos\beta\cos(\beta-2\varepsilon)} = \frac{l\cos^2\varepsilon}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)},$$

so that, at $\beta \to \pi/2$,

$$\Delta x' = \frac{r \sin 2\varepsilon}{\cos\beta\left[\cos\beta\cos 2\varepsilon + \sin\beta\sin 2\varepsilon\right]} \sim \frac{r}{\cos\beta} \to \infty$$

Nevertheless, the projection of l' onto the axis X has the same length for any direction of sight: $l_x = l'\cos\alpha = l$. This roughly describes how the fine details of the picture get lost far from its center: any discrete structure gets eventually absorbed by the focus zone, producing the impression of continuity.

Inside the focus zone, we neglect the difference in the distances of the point of the object line X from the observer: they all belong to the same level of hierarchy. That is why we can model the focus zone by a segment orthogonal to the line of sight. However, the alternative representation with a small arc does not introduce any significant corrections, and the visible size of the focus zone, as the correction

$$\Delta l = 2r(\mathrm{tg}\,\varepsilon - \varepsilon) \sim \tfrac{2}{3} r\varepsilon^3$$

is an infinitesimal of a greater order at small ε (that is, it belongs to a different level of elaboration). The issue of the primacy of the straight line or the arc (sliding movement *vs*. rotation) may be of high importance, say, for the foundations of physics; but here, we talk about something else.

In general, one can focus on an arbitrary point of the object line. The perceptive form will then be represented by a continuum of fragments of different size. For a complete description, a finite number of observation acts is enough, as soon as their respective zones cover the entire span of the object. Every such map reveals a hierarchical structure, arranging the parts of the object by their distance from the observer. There are as much covers like that as you like; still, all of them belong to the same integrity, representing the object as a whole. Such an integrity that can unfold itself into various (though never arbitrary) hierarchical structures is called a *hierarchy*. Perception as an activity is hierarchically organized.

We have demonstrated, how a very primitive mathematical model could lead to the idea of hierarchically structured "inner" space used by the subject to regulate any other activity. More realistic models would account for the "fine structure" of the perceptive act, as general psychology indicates that any action within an activity grows into a hierarchy of operations; for instance, the zone of immediate grasp might be represented by an angular (say, Gaussian) distribution rather than a mere range of angles. In this case the overlap of the zones is to produce an analog of a musical timbre, with the angle of 2ε playing the role of the principal tone. Such an underlying structure implies certain considerations of regularity and stability (robustness) of such "inner timbres": there are those that are better fit for zone discrimination and emergence of a well-defined scale. As we know, in music, a similar approach predicts just a few preferable pitch scales (associated with the base collections of tones, musical modes, or chords); there are only two scales that can pretend to universal significance and recognizably reproduce any musical intonation at all.[9] The same structure has also been empirically discovered in visual arts; however, without any sound reasons for such an eminent resemblance.[10]

To be precise, there is no need to cover all the object with the focus zones; instead of a complete map, one could just take a few reference directions, with all the rest somehow "interpolated". Putting aside any big science, one can easily comprehend that by personal experience. People are apt to judge by a couple (or so) of "typical" traits, never bothering about too thorough investigation. The standard techniques of producing a plot of a function follows the same line: first, we ponder a little over the selection of reference points, then we connect the images of the selected points with a smooth line (a kind of spline). Of course, the quality of the graph will depend on the choice, and on our ability to detect the singularities; in our everyday experience, we sometimes make errors, but life will quickly put things in their proper order.

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[9] L. Avdeev and P. Ivanov, "A Mathematical Model of Scale Perception", *Journal of Moscow Physical Society*, v. 3, pp. 331–353 (1993)

[10] P. Ivanov, "A Hierarchical Theory of Aesthetic Perception: Scales in the Visual Arts", *Leonardo Music Journal*, v. 5, pp. 49-55 (1995)

For one of the most important corollaries of the zone model of perception, one could mention the inner hierarchy of each scale, the presence of "embedded" discrete substructures. In the simplest model, we could arbitrarily change the distance from the object, obtaining different perceptive hierarchies (the collections of maps with varying zoom factor). In a more realistic approach, like the above inner timbre model, the feasible scales are not arbitrary, and we get a number of preferable observation points depending on the nature of each scale.

Earlier, considering the "grazing look" limit $\beta \to \pi / 2$, we, of course, violated the model's applicability conditions. In reality, the possible range of observation angles (the observation field) is determined by the character of activity; it will never be too extended. When the number of focus zones to cover the object is too high, this hints to the necessity of moving to a different level of hierarchy. That is, we feel ourselves as if we were *inside* the object, which results in a split view presenting the whole object as a collection of distinct parts, which can be viewed using the standard active inspection technique. In the opposite limit, with the object entirely covered by a single zone, leads us to the notion of a point, which too requires switching the level: there are no points in real life, and the inability of discern anything means that we just overlook something that could be revealed at a closer examination.

The mathematics of zone perception pictures it as a process unfolding in some global (embedding) space, so that any perception theory could be developed in respect to that formal space. Now, let us put the logic upside down: one might conjecture that the hierarchical vision of space reflects its true nature, while any abstract absolutes only refer to a specific level of hierarchy, a range of prototype activities. Such a position may superficially seem a regression to the primitive anthropocentric view of the world that was so hard to get rid of in the history of science (and some science is still too anthropomorphic to be true). Admitting that the form of an object could depend on the way of viewing looks like stuffing the theory with unworldly subjectivity, mystics and voluntarism. But does anything in this approach specifically refer to the subject? We rather discuss the overall organization of the system, hierarchical structures. Any natural things interact on many levels, and the appearance of one thing for another depends on the nature of interaction, which does not need to be related to conscious activity: any physical phenomenon, or biological metabolism, will do as well. Let a robot view the same object instead of a conscious person; the machine will reconstruct the same image, as soon as the viewing technology is hierarchically structured. An electron will be a hierarchy for another electron, provided the act of interaction admits inner structure and gradual development. In physics, theoreticians often employ a kind of smooth introduction or removal of interaction ("turning on and off"). However, they usually take it for an auxiliary idea, a technical trick, making their best to remove any scaling parameters from the final result. Don't they suppress the physical sense (the purpose, the object area) of the theory that way? Abstract theories are deemed to describe anything at all; this is the principal source of formal and logical contradictions. Just take the relativity theory: first, we limit the propagation speed of any physical interactions, and right away, treat space-time as global and ready-made; isn't it crooked? To be consistent, we need to consider the process of reference frame formation, without too precocious assumptions about what will happen in the vicinity of the horizon (see the above note on the grazing-look limit), or beyond it. Well, such a theory would not show off spectacular singularities to collect easy funding. Instead, it would be a science about nature, and not about an abstraction that we pretend to pose as nature. The hierarchical approach, therefore, is to overcome the anthropocentric prejudice in a most drastic manner, while the formalistic (objectless) science tends to fall into all-embracing subjectivism. In the zone hierarchy vision, the existence of an embedding space means a fixed scale as a specific level of perception. For each scale, there is a objective hierarchy of embedded scales; with the transition to another scale, the whole construction will change, up to an unrecognizable figure of the same object. As long as we stay within a definite level, we are bound to get numerous "illusions" and "artefacts"; still, even those are not arbitrary, since they grow from the organization of that particular scale and hence are quite real in the course of the corresponding activity. Logically, this does not mean that we can identify such "inertial forces" with real, "physical" phenomena; nevertheless, in any activity, we have to account for our place in the Universe and act in accordance with our (albeit biased) vision of the world.

And now, the blunt question: does that have anything to do with math?

Yes, it does. The attempts of some mathematicians to oppose their science the rest of the human culture, to put themselves "above" nature and dictate their "absolute" truths to it, are of the same train as the attempts of the "deepest" physics to speak of the world in general rather than the small part we happen to face so far. Once having chosen a very special mode of perception, one tends to forget about all the other possibilities, the multifaceted nature that can be treated in many different (and not always compatible) ways. Within the limits of its applicability, such a castrated science is quite efficient and useful; as soon as it comes to assimilating complementary views, it becomes a handicap. Of course, no formality can stop life: it will penetrate mathematics through the heaps of concealment and vague assumptions, which are not to be exposed by the "rigorous" science to the wide public. Later on, somebody promotes a new paradigm, showing the bulk of the former theories in a brighter light, which allows to "naturally" unify them into a flash of beauty. A storm of applause and loud cheers: great! Presumably, that genius did not do anything special; it's mainly a matter of recollection of what the others have long since preferred to forget.

Elementary Sets

Most of get acquainted with elements and sets yet in the elementary school, and some may have touched the topic even earlier. As a bit of science still happens to come handy from time to time, the thought of a set seems to be a natural background for further research, before any formalization at all and as its indispensable condition. Something belongs to somebody; some may possess just nothing. This is what we see anywhere around every single day. Well, lets us call a piece of somebody's belongings an element, while the owner of the thing will take the name of a set. It may come up that the presumed owner, in its turn, belongs to somebody else. Attempting to sort that out, we would keep all those who are born to possess within one class, putting those who are made to belong into another class. That is, in addition to sets, there also exist classes, which resemble sets in many respects, but not entirely.

The way we work with sets and their elements is in no way spun out of thin air; it simply mimics the social order dragged from one millennium to another. In sets theory, mathematics is a mere reformulation of (bourgeois) sociology with its principal question: equal or stranger? The attitudes depend on class (estate, race, ethnic, civil, corporate, clan, group) membership. That is why it is so important to decide first on whether the element belongs to a set; the details of the inner structure of the set are in the second line. Obviously, life embraces many things that do not fit in the set-theoretic approach; still, as soon as we have postulated the primacy of the black-and-white logic, the inconvenient questions may well be swept off to a dusty yard side, so that some applied science could take them out for a momentary need, dress up in mathematical gowns and exclaim: how pretty! though the formalities get soon abandoned for something less respectable.

From the set's viewpoint, all its elements are all the same. Of course, they are well distinguishable (otherwise, how could we speak of a set?); still, for some higher judgment, their distinctions just do not matter. The elements, however, may fail to accept such a uniformity: they have something to divide, and some are eager to get more than the others. That is, with all the qualitative homogeneity, there are quantitative differences. The law is the same for all; however, those very big may bluntly scorn it while dealing with the small fry (still demonstrating a pious respect in the relations with those as massive). One way or another, the problem of the difference of equals keeps on, and every generation has to treat it their own manner.

A mathematician goes straight on: each element can be loaded with a number representing the "social weight" of the element, its importance for the whole. If, for example, having a set of two elements {*elephant*, *banana*}, we are curious of how many elephants and bananas we really have, and if we find that our bananas much overweigh our elephants, we refer to that set as a set of bananas with an admixture of elephants; conversely, a set with a prevalence of elephants will be called a set of elephants with an admixture of bananas. This exactly corresponds to how the evaluation of people by the size of their capital brings us to the conclusion that this world is made for the rich, while the poor are nothing but a regrettable lapse of nature, a source of petty nuisance. Ideally, one should draw all the poor into some reservation, thus putting them in an entirely different set. Such an exploit, however often undertaken by the political elite, has not yet once succeeded; this circumstance is specifically reflected in mathematics as well.

Why not? Nobody can prevent us from formally representing the community of elephants with banana (so to say, "a mixed state") as a union of two "pure" (one-element) sets:

$$\{elephant, banana\} = \{elephant\} \cup \{banana\}$$

Does that change anything? Nothing, for a superficial eye. However, recalling that an element's belonging to a set is not a heavenly revelation, but rather a statement allowing a practical justification, we observe a very important discrepancy between the two parts of this (meta)equation: in the left-hand side, it is assumed that there is a *uniform* procedure for establishing the fact of belonging regardless of the quality of each element; on the contrary, in the union representation, two *different* procedures are assumed, and being an elephant manifests itself in a way other than being a banana; finally, identifying the left-hand and right-hand sides of the equation we make a very strong assumption that one activity will always result in the same outcome that a quite different activity which has little in common with the first. An ocean of question: what does it mean, "a result", "the same", "always"? Different answers, different theories.

Thus, it may happen that the difference of elephants and bananas only exits in the context of the set {*elephant*, *banana*}, while an elephant on itself and a banana on itself are undefinable. In a class society, situations like that are quite common: can there be a slave-owner without slaves? a capitalist without wage labor? Well, mathematicians have invented a standard work-around: instead of a set like {*elephant*, *banana*}, just take a cortege ⟨*elephant*, *banana*⟩, which then can be skillfully manipulated into an "ordered set"; as an immediate consequence, we find that the idea of "order" is to complement the idea of "set", which opens a vast area for proliferating entities up to complete chaos, thus extinguishing any glimpse of reason.

A professional would indignantly repudiate such silly suspicions. Isn't it obvious that a set of two elements can be easily ordered in no time? it is enough to indicate, which element is to be treated as the first. With this consideration, the cortege ⟨*elephant*, *banana*⟩ is just a conventional contraction for the set-language record {*elephant*, {*elephant*, *banana*}}.

No, this is not as obvious. What do you mean by "indicating"? There are millions of variants. For instance, we could as well put forth the element with the maximum weight, and order the set in direction from the most "massive" elements to negligible contributions. The cheat trick is in the silent omission of the fact that the set {*elephant*, *banana*} is indeed a set, and, as such, it cannot directly become any other set's element. To be honest, we should speak about the set {*elephant*, *X*}, where *X* is, in some sense, equal to the set {*elephant*, *banana*}. At this point, any dummy will see that there are so many senses, and the "set-theoretic" definition of a cortege is entirely dependent on how we choose to reduce the complex entities (sets) to simpler entities (elements).

Even for one-element sets, the element *elephant* is quite different from the set {*elephant*}. What can be said about sets does not pertain to elements, and the other way round. For instance, we define the cardinality of a set; in classical theory, the cardinality of an element is a nonsense, while the cardinality of a one-element set is unity (by definition). Elements can belong to sets; the maximum what a set can afford is to be a subset (with many pitfalls, here too).

In our common life, we can easily consider the same thing in different aspects, depending on what we are going to do with it. A book can be for reading; but it can also serve as a prop for something, or become a kindling stuff; a book can incite hostility, or, conversely, be a symbol of unity… In the same manner, anything can be treated sometimes as a set, and sometimes as an element; but these still remain two *different* treatments! An arbitrary mix of the both in the same discourse would be a logical fallacy. Over and over, we meet the same problem: what is "the same"? No formal theory can answer; this is an entirely practical decision. One could easily observe that, with any idea of the unity, the elementary and set qualities will be its different aspects, the distinct modes of usage. Such partial manifestations may be almost independent of each other (thus, one could be a good dancer, but also a poor husband); in reality, we much more often encounter the opposites that are essentially interconnected, up to being utterly incompatible, so that defining one of them, we also define its counterpart. What is taken for an element is not a set; what is a set cannot be an element.

Can we fancy something that would be both an element and a set? Yes, we can. But that would lead us beyond the limits of the classical set theory to consider all kinds of the superposition of "pure" sets, or some analogs of the quantum-mechanical density matrix. That may open a most promising direction of research, a different science.

Meanwhile, let's turn once again to the sets whose elements, beside the qualitative definiteness, may also have a quantitative load, a weight. It might look like a trivial generalization of the classical set, where the possible values of the weights are restricted to zero and unity. Indeed, let us admit that some elements exist as several instances… To exhaust a set, we just blind-hand rummage in a set and pick some of its elements; let it be WR sampling well known in mathematical statistics, or repetitive enumeration like in computer programming (with its bag objects and various iterators). Nothing new.

But hold! Sets with repetitive elements essentially differ from "weighted" sets: in the former case, we deal with the classes of equivalence; in the latter, with the intensity of the presence of each element that cannot be divided into separate instances. An exhaustive enumeration of the first kind will give the complete number of all varieties (with an exact quantity for each variety); the enumeration of a "weighted" set will give each element just once, but with a certain weight (in one of the possible modifications). Classical and quantum statistics differ in a similar manner.

Well, this too does not seem to be much of a news. We have long since invented the theory of fuzzy sets (though, in fact, it still remains a good intention rather than true theory; a template for special implementations depending on the techniques of combining the membership functions). The idea is really great: an element does not entirely belong to a set, but rather tends to belong; in particular, one comes to distinguishing the volume of a set (the number of elements) and its mass (the sum of the weights). In principle, the weights can evaluate to any real numbers; however, we are free to normalize them to the total mass to bring the weights to the interval (0, 1), or, alternatively, divide the weights by the number of elements, which means a transition form extensive quantities to densities (so handy for infinite sets). The wider conceptual choice, the better for applications.

Now, what is missing in the soup? Just one thing, science proper. Those who have learned to press the computer keys or tap the screen cannot yet be considered as IT gurus, or at least power users. The monkey mode of working with sets does not make the matter a trifle clearer: we still do not know why we should act that very way, and where we'd better seek for a different toolkit. To spread like must over a volcano is far from specifically human ways. Reason is to make the knowledge of a science's limitations grow along with that very science.

Mathematicians are apt to believe that formal constructions exist as they are, given to the humanity from a superior realm, so that the earthly things can only implement the universal ideas in a random and incomplete manner. Hence the usual terminological confusion: in mathematics, the term "model" is used to denote special implementations of the abstract schemes, while reality is exactly the opposite: mathematical theories are nothing but vague, one-sided, approximate reflection of certain aspects of human activity; one need to add much and much to mathematics, to make a formal carcass of a real thing just a little bit meaningful.

Most often, formal knowledge is just a kind of covertly acknowledging one's ignorance: we know as much as nothing, but we can well make up for it blowing the cheeks and pretending to be wise. Let a stranger ask: why at all? and there is a prompt answer: this is an old and respectable tradition! We are used to do that, and those who dislike it may proceed their own way, we don't care. Well, the mere fact of a conscious choice of the mode of action is already a good sign. However, there is nothing bright in stretching (or cutting) the world to habitual standards. As the world would actively object to such a treatment, the choice turns into self-isolation, rejection of anything that is not our way: true science is just what we do, and those who think differently cannot enter the academic circles (with the appropriate organizational and financial consequences).

Science begins with trying to name everything, to move from working with things to playing with their abstract representatives, terms and formulas. It's quite normal; however, science should not stop at that. For example, in medicine, many diseases are referred to by the name of the organ that does not properly work in some respect: gastritis, bronchitis, periodontitis, sinusitis *etc.*; the next step is to find out what exactly goes wrong, to be able to further ask why. Mathematics is primarily a universal technology of making names. What stands behind the names is to be clarified at the next stage, with more appropriate methods.

Inventing a name is in no way going to augment knowledge. Naming is an important preliminary phase of cognition, a kind of a declaration of intent: we have noticed some peculiarity in the world, and we are to look closer and dig deeper. All right, we can fiddle about with sets; now, let us pay attention to what we really do.

In the general philosophical plan, each conscious activity has its object and product, and the subject of activity is to transform the former to the latter. In any particular activity, nature is first perceived as a chaos of things and events, the possible objects. We need to select from this mishmash something that could help us in pursuing our goals. Each dish requires certain ingredients; we seek for them in our environment, ticking off the needs that are already satisfied. When all the boxes are checked, we can start working, as we have at last the specific object of that very activity, the raw material for the intended product. Such objects are represented in mathematics by sets.

This already implies much. To become an element of a set, a thing must feature quite definite (in the current context) properties, that is, match the industrial standards. When a taskmaster makes a call to the construction site asking whether they have nails, he does not needs a formal answer (yes, I still have a couple in my pocket) but rather an estimate of the sufficiency of the available stock for a particular task. In other words, the membership of an element in a set means the completion of one operation (acquisition) and switching to another (production proper).

Further, since the list of ingredients is determined by the end product rather than the conditions and circumstances of activity, sets can be constructed in many different ways. In the simplest case, we have something like a number of measures that are to be filled with the appropriate matters (one egg, 200g of flour, 100g of sugar, a pinch of salt, baking powder on a knife's tip, 1/4 glass of cream). When we fail to meet a particular request, we can use something of the kind, respectively changing the proportions. That is, each product determines a number of sets rather than a single set; this is a set-theoretic universe with a structure adapted to (and formally studied within) a specific problem. Instead of a single set theory fashioned once and forever, there is a bundle of set theories induced by special formal models (mathematical products). The degree of resemblance of one such theory to another is in no way related to their inner virtues; it comes from the similarity of the parent activities.

Philosophy indicates that an activity (as a manifestation of reason) cannot be a mere incident, a momentary and unique act; any activity is a cultural phenomenon implying regular reproduction of some product and the social conditions required. As a consequence, the phases of production preparation and production as such do not need to follow each other in a strict order: in the cyclic reproduction they become relatively independent, so that we can just accumulate everything needed for production and spend a part of the stock when all the prerequisites are in place; the rest is to be passed to the next production cycle. This exactly what the mathematical abstraction of a "weighted" set means: a "store", a number of cells for raw materials, plus the level of occupation for each cell. Quite naturally, a lot of possible implementations. Thus, when the containers are vast enough and the ingredients do not significantly interact with each other, one cell can contain as many items as we like; in the opposite case, one cell can contain just a single item (or be empty). This is the way the "boson" sets differ from "fermion" sets (the latter constitute the realm of the classical set theory). And, of course, any conceivable combinations.

An important margin note: the quantitative differences of the elements are related to reproduction cycles; in a static theory, they represent time. Mathematics will always incorporate these two aspects:

qualitative definiteness (spatiality) and enumeration (sequence, ordering, as an abstraction of time). One can be expressed through another as largely as we like; this won't remove the very opposition, rather shifting it elsewhere, to some hidden suppositions. The unity of space and time is only possible in a special activity that, for our purpose, could be called measurement.

All of a sudden, our modest industrial warehouse would grow into a full-fledged space: a spatial dimension corresponds to each element, while the weights of the elements turn into spatial extent along the respective axis. In this picture, the traditional, classical sets are represented by various hypercubes; less trivial constructions may refer to spatial areas of an arbitrary form. For an alternative, one could prefer labeling classical sets by the points of such configuration space; this gives room for other generalizations: for instance, we could be interested in the dynamics of transforming one set into another, of course, within the same matrix activity, in respect to its product.

That is not yet the whole story. Classical generalizations of traditional sets assume a quite definite mode of establishing an element's membership in a set: if we put something in, it is expected to stay there for all times (or, at least, until the next scheduled inventory). In other words, the elements of a set are believed to be well isolated from each other, they never react. Which is certainly not the common case. Real things go off, produce other things, change location, merge and split. Put dough in a stove, and take bread out; plant a seed, and get a prolific tree. Additionally, the very way of extracting a thing from the stock may involve various transformations: thus, having bank accounts in US dollars, euros and Russian rubles, we may wish to get some Suisse francs in cash. This is how quantum sets are born.

Specifically, a single numeric estimate of an element's membership in a set gets split into complimentary parts, just like quantum mechanics begins with replacing real probabilities with complex "amplitudes". In Dirac notation, the possible incoming elements are represented by the vectors of some configuration space $\left|\alpha\right\rangle$, while the available outcome of any sampling is described by the functionals $\left\langle\mu\right|$, so that obtaining the "membership function" μ for a set in the state α is related to the transition amplitude $\left\langle\mu|\alpha\right\rangle$. The inherent activity-framed character of a set is vividly stressed in the so called second quantization formalism, where adding an item a means application of a creation operator a^+, while removal of an item form the stock is related to an annihilation operator a^-. Just like in quantum mechanics, the operators may be noncommutative, so that the structure of the set would depend on the process of its construction. When mathematicians boldly identify the objects obtained in different ways (2 = 1+1 = 3–1 = 12/6), they implicitly presume the existence of some activity involving all such products on the same footing; there is no random choice or arbitrariness, this is an entirely practical issue. Without such a cultural background (albeit in the form of children's play or abstract curiosity), there is no science at all, but rather (in the most innocent case) an art of (symbol or public opinion) manipulation.

Now, from the viewpoint of the matrix activity, a set is just a number of tags, or labels, that we stick onto anything at all that might have some (however distant) relation to the job. In psychology, such a categorization, grouping the observed phenomena according to pre-determined criteria, is known as perception, which differs from mere sensation by that very universal activity of triage. Sensation would produce an image of a thing as it is, and we (as any other living creature) are made so that such an adequate reflection were possible, of course, within certain limits. Sensory pictures may be insufficient, but they never fool us. On the contrary, perception gives the image of a thing as we picture it to ourselves; here is a source of illusions and mistakes. Historical development, in this respect, is to compile all kinds of conceptions suited each for its specific task. Ideally, in any activity, we should employ the gauge that would reproduce the organization of the activity as neatly as possible. In real life, any correspondence is but approximate; still, some choice of a reference frame is inevitable, since there is no other way to outline the object of activity in the infinitely diverse world. In psychology, the perceptive scales like that are called sets; the same holds in mathematics.

In this context, it is evident that the formal constructs like an "empty set" or the "set of all sets" cannot be proper sets; they belong to other levels of activity and their object are is different. Our perceptions are culturally determined; hence our conceptions are not arbitrary, they always reflect the already available practical options. Conversely, each object area is culturally linked to certain types of activity. With all that, the idea if an empty set may refer to the object area, while the hierarchy of the possible scales provides a common universe embracing sets as its "members"; still, a set as an "element" of a universe is not the same as that very set as a collection od elements, and it would be logically incorrect, to mix the two connotations in the same formal context.

The notion of set may seem to be more general than the notion of a "weighted" set of any kind: in formal consideration, we apparently "superimpose" additional characteristics on a carrier set, just attaching the weights (population numbers) to the original elements. However, the same formal approach would say that, conversely, a set is a special case of a "bag", one of its possible projections. The right answer is that the both abstractions reflect certain levels of activity; in every hierarchy the order of levels is relative, it depends on the way of unfolding. No doubt, in some cases, it is quite enough to think of a set as a collection of boxes that can be filled or emptied. Reality is often different, and the structure of the "store" may follow the evolution of the industry, adapting to the current needs. For example, take the different classes of cargo vessels: tankers, bulk carriers, container ship *etc*.; similarly, the invention of the virtual circulating media will significantly change the currency market. We are free to recourse to any abstractions, provided we do not forget about their limited area of applicability. The interrelations between abstractions never refer to inherent superiority of some of them above the others, but rather to their common origin from something that cannot fit in any abstraction at all.

In the spatial picture, a union of sets corresponds to an increase of dimensionality; the intersection cuts out a common subspace. Still, from the objective viewpoint, in both cases we speak of kindred activities with a common object area. However, such a commonality can be sought for in two opposite directions: either this is an extensive search accumulating everything that has ever been used for something, or one could seek for universality, selecting what might feed many similar products. One always goes with another: many different things must be present to reveal kinship, and there is no distinction but within a certain commonality. Formal science is often seduced by the mirage of abstract universality: just find the most fundamental building blocks of the whole world, and there would be nothing to wish for, and the humanity would cut any worries and rest on the laurels… However life punished people for that lazy complacency, some scientists still believe in the final law of all that happens: well, in the early epochs, we made lots of mistakes, we did not properly understand, but now, with all the fruits of the progress at hand… Eventually, this does not differ from religion, up to the sign flip. Real world is subject to change, and not only due to spontaneous (background) evolution, but mainly because of our conscious interference, in the course of cultivating and intentional reformation. Every now and then, we put ourselves in a different environment carrying the traces of our influence on natural things; this adds yet another level of mediation between the properties of things as they are and their cultural involvement.

New technologies lead to new sets, be it aircraft construction materials, or kitchen stuff, or inhabitable lands. Climatic shifts will modify the meaning of the seasons and their duration. Some of the terrestrial languages are bound to die out, and some new languages are yet to come. Similarly, the units of measurement in physics and engineering get adapted to the new variation range. It is most unlikely that mathematics could always stick to the millennia-old mental framework.

Nevertheless, many cultural phenomena persist for long, compared to the average duration of the human life, or even to the span of a single historical epoch. Within that period of stability, as long as the overall character of activity remains the same, there is no sense in proliferating theories; any observable modernization is a manifestation of a hidden social need and a presage of a revolution. Of course, nobody can just cancel earlier habits; they will coexist with the new trends in several generations.

Universal sets can only exist as a complement of non-universal sets, which, in their turn are not universal only in respect to some earlier established universality. Under certain conditions, one opposite can easily transform into another. This is what we call hierarchical conversion.

There is a popular candidate to the place of a universal form: language. Apparently, as long as we restrict ourselves with the cognitive aspect, language is a true embodiment of the idea of categorization:

every distinction has a name, so that the vocabulary becomes a natural universe for everything. Yes, the vocabulary is bound to expand; but these are regular modifications that are quite compatible with the ideal of a super-dictionary containing anything at all, and serving as a natural limit for every linguistic development. The next step is to canonize the alphabet; the makes the limit an absolute common for all minds, and the academic nirvana seems not so far away…

This line of thought piles one illusion upon another. Language cannot evolve in a purely quantitative manner; primarily, its development means a change in its content. Even within mathematics, huge conceptual changes have occurred in just a few centuries; as a result, old texts in the modern interpretation may be far from the author really meant. The same holds for the presentation techniques, the "alphabet": the zoo of mathematical notation does not fit in Unicode, virtually loosing the very sequential appearance and employing many-dimensional diagrams that cannot be reduced to the traditional discourse but in a few very special cases. The words "number", "space", "function", "truth", and of course "set", get re-interpreted by every generation of mathematicians in comparison to the terminological newcomers like "algorithm", "applicator", "topos", or "fractal".

Taken in different proportions, the elements of a set determine different products; all that unfolds on the base of a definite technology, so that the possible component structures would lead to compatible (or comparable) products. To pass to an entirely different class of products, the very object of activity is to be changed; in general, such objects do not depend on each other: to allow for their natural union, we need an activity somehow embracing them all, with all their specificity. That is, some activities may implement an activity of a "higher level" (see above about the relativity of such an ordering). The hierarchy of sets is to formally reproduce the hierarchy of activity. In the simplest case, we get a tree-like structure: food industry is pictured as a number of relatively independent branches like agriculture, dairy industry, fishery, cattle breeding… Om the other hand, any recipe would combine quite different ingredients like flour, eggs, cabbage or meat; with al that, bread is something different from a fish soup or an ice-cream dessert. This, once again, illustrates the flexibility of hierarchies, heir ability to unfold in many hierarchical structures without loosing the qualitative definiteness. Nothing to say about more intricate inter-level relations far from a trivial tree.

With all these considerations, set theory (elementary or not) can be explored in a cultural context permitting most diverse generalizations. This is the only way to make mathematics truly meaningful. Everybody is free to play with forms; moreover, practical applications do not necessarily involve sheer material interests, as we can be inspired by beauty, be systematic, or dream of the future. Any need demands certain instrumentation, and formal instruments in particular. Because of the unity of the human culture (as an expression of the unity of the world), no theory, however weird, can grow from nothing or be utterly useless. Still, an extra bit of reason in our attitude toward our creativity will hardly do any harm.

Rational Dimensions

Basically, the world is quite simple. It is cumbrous specifications that make it complex, as we try to discern ever finer details. Everything would seem equally valuable, and one would struggle to keep on a slightest bit of the whole. This spurious hope leads from one specter to another, so that, in the end, it's becoming hard to believe in anything but apparitions. Science gets bogged down in the speculations of scientific method, art is to incessantly savor a fashionable caprice, philosophy tends to force nature into a single all-embracing scheme. In the moments like that, it is especially important to hold back for a while, to drop a far-away look at the world and our worldly destiny. It often happens that more detail does not automatically imply higher accuracy, and popular explanations may be far from vulgarity. Hundreds of formulas cannot eclipse the effect of an apt figure, while a rigorous theory may never go farther than an extended metaphor. A few naive observations to please a curious amateur may well come as imminent in science as thorough calculations or laborious experimenting. Well, those seeking for provocative originality are free to skip the rest of the reading.

In real life, we are accustomed to move in any direction, combine numerous experiences, discover unexpected faces of the quite ordinary things. On the other hand, to cope with an intricate undertaking, one may need to split it into distinct stages, fix intermediate goals, or run branch projects. In either case, there is a wide field for cooperation, with the whole activity distributed among several (sometimes many) individual participants. Putting themselves to science, people would build it reproducing the very same logic: any research is aimed at the enhancement of either our ability of combining the building blocks or the techniques of disassembling things into potentially reusable parts. No need to explain that the requirements of production always dominate over inquisitive analysis, so that any decomposition would serve the interests of some future construction.

As soon as it comes to connecting separate things or breaking the whole into elemental pieces, there is a question of the possibility of universal methods, or, so to say, master technologies. Philosophy says: yes, they do exist; however, there are no pre-defined patterns, or assembly charts, but rather specialized implementations of the same principles depending on the current instant and need. Such fundamental ideas are known as philosophical categories. Of course categories may be built into various categorical schemes, or represented by a hierarchy of schemes.

The antithesis and mutual penetration of quality and quantity is probably the most famous scheme in the cultural history of humanity. So, let us take it for the starting point of our review of the mathematics of dimensionality. Once again, without comprehensive elaboration, in general, in an intuitive manner.

Admit that we keep producing a socially valuable something. Provided the product satisfies the corresponding cultural need, we find its quality quite acceptable. Any other product that might satisfy the same need (within the same activity) will do as well, being of the same quality. In this sense all such products are equivalent (interchangeable). Still, the very “otherness” assumes distinction; this difference of the qualitatively identical things is called quantitative.

In this context, quality and quantity are interrelated entirely within the embracing activity, in respect to its product. Certainly, similar relations also exist in inanimate or living nature, regardless of human intervention. Still, in a note devoted to formal space construction, we’ll stick to the activity prototype, with other object areas to be discussed in a different wording.

Obviously, mere distinctness of equal things is not yet a kind of quantity. Thus, a Chukchi reindeer breeder may know each of his animals by name and aspect, so that, for him, they are all qualitatively different, though one reindeer is as good in a sledge team as any other. In this manner, quality can easily grow into a hierarchy; for humans, this is related to the interdependence of different activities. Similarly, in mathematics, the elements of a set are in no way dependable on each other, they are just different, but any single element belongs to the same set. On the top level of the hierarchy, speaking of sets, we do not pay much attention to how one element differs from another; this is a lower-level quality.

To make the difference quantitative, one needs, beside production and consumption, to bring in some order. That is, all thing satisfying the same need are to sorted out, with establishing a gradation in priority and preference. In the limit case, when one thing is absolutely no better than another, such an ordering may develop from some outer influences: this thing was made before that, or maybe just came to sight first… However arbitrary the enumeration may seem, some practical considerations can always be found. The position of a thing in the row represents its quantity, in the narrow sense. Of course, quantity can also be hierarchically organized: at some level of preference, one finds a group of things, so that additional effort is needed to establish order within the group. From the higher-level viewpoint, such inner quantitative differences will look as negligible, or even infinitesimal corrections. Additionally, the cost of separation of one level of adequacy from another may vary; thence yet another quantitative hierarchy.

In the theory of dimensionality, quality and quantity thus interrelated in the context of a common activity represent space and time respectively; together, they form what we call a space on unit dimensionality (one-dimensional space, or spatial dimension).

To put it plain, the term “space” expresses the possibility of motion, going from one possibility to another “in a natural way”, following a once established order. To ensure that our abstract space represent a portion of a real world, we need to order things in a right (objective) way. Space and time are just one of the aspects of motion; they are only definable in respect to a certain class of changes.

Qualities are never comparable as such. Two qualitatively different thing are just different. However, the presence of order makes thing comparable within the same quality. We speak of one as closer, another as farther away… As for equality, it is not as straightforward; basically, it is an indication of an inner hierarchy.

Let us adopt, for a while, the traditional "plain" digitalization of a one-dimensional space: there is the origin of the system (a reference point related, for instance, to the subjectively ideal product of activity), and any other point is characterized with a number generally meaning the time needed for the transition from the origin with some "standard" speed; in physics, following the ancient tradition, the speed of light is taken for the standard, and that is why ("by construction") this speed is the same in any frame of reference.

One can easily guess that there are no negative quantities in nature. Any hierarchy unfolds itself from the top down. Still, comparison of positive numbers provides a hint to the direction that should be taken to come to the destination point from some initial position; this direction we conventionally designate with the sign + or – . In more complex (multidimensional) spaces, the direction will be represented by the phase (angle), or something like that. In general, the transition will follow one of the possible trajectories. In everyday life, this corresponds to the choice of a particular mode of production, unfolding the activity into a hierarchy of actions and operations. For instance, to get a warehouse, we can build the edifice from scratch; alternatively, we might choose to rebuild some church to serve the purpose.

Depending on what and how we enumerate, all kinds of space-time will arise. In the background, there is certain activity, whose organization our mathematics is to reproduce. Discreetness and continuity, finiteness and infinity, no borders or some limitations… These are the examples of qualitatively different quantities. The conversion of quantity into quality is the other side of the same. Indeed, chairs may vary in size; however, a very small chair is no longer a piece of furniture, but rather a toy; similarly, a giant chair is alright as a sample of monumental art, but nobody is going to sit there the common (regular) way.

To be sure, nature does not know any zeros or infinities. Everything is good to certain extent; lack of moderation often leads to regrettable consequences. As long as we merely toss abstractions, there is practically no risk: well, yet another paradox, or an intrinsic contradiction… Just get into real-world production, and the abstractions will need to become workable tools, their robustness to be proved by the typical applications. In many cases, within each quality, there is vast zone of admissible quantitative variations, far from the limits of applicability; this is a kind of operating range where any border effects are of no importance. The commonly known idea of a spatial dimension as the possibility of infinite motion in any direction is an example of such a local structure. Centuries ago, this extensibility was meant by default. Today, everybody heard about quantum mechanics, which is primarily focused on the study of boundaries, admitting any inner (local) motion as virtual, where appropriate. Still, a certain degree of moderation would be welcome, there too.

We have already seen that quantitative and qualitative hierarchies can be unfolded in many ways. On the preliminary (planning) stage, all kinds of weird ideas are allowed, all the visions of the product, and we intentionally loosen the hold of our fantasy in order to grope for the right direction of development. Finally, it is time to start: at a certain moment we decide that the construction is acceptable as it is, and stop any further elaboration. From now on, one can deal with the structures in the science-like manner, leaving aside the philosophical questions. It is only encountering a nonsense that will revive the interest to the foundations.

Admit that we are given two one-dimensional spaces. Is it always possible to paste them together into a two-dimensional something? From the formal viewpoint, nothing prevents us from considering a pair of numbers, each one for the corresponding dimension. Two pounds of pork, and a stone block. Since all aspects of human activity are heavily intertwined, there is always a possibility to discover (or intentionally produce) the situation where such a fusion would not seem a bit surrealistic. Still, even there, we intuitively feel that mere concoction is not enough for the feel of spatiality. Two qualities could only be considered as the different dimensions of the common space if there were a qualitative homogeneity, a kinship: though each dimension represents a separate quality, each of them also represents the same a higher-level quality. In other words, there is an activity that assumes a combination of the two other activities, each of the two being equally necessary for the final product. The mode of aggregation does not much matter. Thus, we can first set up a fence, and then paint it red; alternatively, we can paint the parts of the future fence beforehand, just supplying them with the rigs for assembling the whole in-place; any intermediate options are also possible (for instance, involving additional paintwork on the seams). The product of such a compound activity can be logically represented with as point in a two-dimensional space, while the process of production corresponds to one of the possible trajectories from the initial to final state.

This is exactly what we do in linear algebra: there are basis vectors, and a sequence of their linear combinations arranged in time. After all, the construction of a basis is mainly nothing but the matter of taste, so that the destination point remains reachable with any choice of the axes and scales. Here, once again, we get an example of the numerous modes of unfolding in any hierarchy, which reveals its different positions (hierarchical structures). The integrity and qualitative definiteness of the product is in the core of that mutability. It is the possibility of rearrangement, the symmetries of the whole that distinguish a multidimensional space from a mere list of assorted qualities ("a tuple"). The traditional notation of the linear algebra id to stress that circumstance: we speak of the addition of vectors rather than component-wise processing.

The omission of the requirement of hierarchical integrity would lead to a mathematical object of a different kind, a graded set, a collection: each element of the collection (a special quality) is associated with a number, which could be interpreted as the prominence of the corresponding quality, or the specific weight of the corresponding component. For instance, a traveler's bag may contain two pairs of pants, a few shirts, several pairs of socks, plus books, medicines, and personal hygienic articles. Taken together, all these things, too, constitute a kind of integrity (the baggage); but this does not need to have anything in common with the characteristic integrity of space-time.

Surely, there is no rigid delimitation. Collections may grow into spaces, as we reduce the independent qualities to a single one (just like market economy is only concerned with the exchange value of the product); conversely, the so called phase spaces (with heterogeneous quantities mapped along different axes) essentially resemble weighed sets.

A multidimensional space represents a quality as expressed trough some specific, or "partial" qualities. In this hierarchical structure quantity becomes hierarchical as well: the quantity of the whole is derived from the quantities for individual dimensions. Practically, this means that the components of the whole will no longer be independent; that is, there is a physical constraint on the system's motion favoring a particular class of displacements. Thus, our habitual Euclidean space relates the square of a vector's length to the squared lengths of the vector's projections to the axes of a coordinate system. Similarly, an estimate of the bank capital will depend on currency exchange rates or share quotations. Besides, the quality of multidimensionality is complemented with a peculiar quantity expressed by the number and sequential order of unit dimensions (from a trivial enumeration of the basis vectors to an outline of economic priorities). The both characteristics are related to the local (metrical) and global (topological) properties of the space. In the following, let us focus on the number of units, the dimensionality.

Now, to construct a multidimensional space, one needs to list the unit dimensions in a certain order, and then relate the quantitative parameters of any element of the space with its unit measures, the quantities pertaining to each component space. A symmetrical constraint will induce the corresponding symmetry in the whole space. In particular, the Euclidean metric invariant in respect to any reflections, shifts, and rotations. The dimensionality of the whole space can be quantitatively characterized by the number of dimensions (the size of any acceptable "basis"). It should be stressed once again, that this picture is only valid in a local sense, within the operating range of the space-time model; for very large or very small "distances", the same constraints are no longer applicable, with quantity becoming quality, and the other way round.

Well, supposing that there are no more significant problems in constructing spaces of any positive integer dimension, can we apply this constructive ability to the unit dimensions of a multidimensional space, representing them with multidimensional spaces of a different kind? Why not?

From this angle, it is the right time to observe that, in any two-level structure (like the whole space *vs*. individual unit spaces), it is not only the top-level quality that becomes hierarchical (and "split" into component qualities), but also the quality of each unit dimension will no longer be the same as if we were only interested in the production of that partial thing as such. The imprint of the whole on its parts and elements tends to distort their original nature, up to radically changing the very their essence. Thus, trying to satisfy the people's need in a specific thing is almost incompatible with market-oriented production, with the same thing to be just monetized; in the latter case, market economy opens much room for half-baked rubbish, all kinds of fake, and even irreparable harm. In mathematics, it rarely comes to the tragic heights; nevertheless, we readily discover an additional level in the quality of each unit dimension, namely, its capacity of playing the role of the dimension number such-and-such in the space of this-or-that dimensionality. In the same manner, the unit quantity becomes hierarchical, with the coordinate value linked to its position in the dimension list. Hierarchical structures of individual dimensions and the hierarchical structure of the whole space are the two complementary representations of the same, the different positions of the hierarchy.

We agree that the relation of the whole space to its unit dimensions is characterized by dimensionality N, the number of components. Conversely, the relation of each component to the whole can logically be expressed with the number $1 / N$ representing the dimensionality of the corresponding subspace. An isolated space may have integer dimensionality M; taken as a subspace of a space of N dimensions, it will be characterized by dimensionality M / N. This is how we come to the idea of rational dimensions.

Let us look at it from opposite direction. Admit that there is an N-dimensional space. Every unit dimension (in any basis) taken as a subspace will have dimensionality $1 / N$. Now, let each unit dimension is unfolded into a space of dimensionality M. Then its quality related to both the embedding space and the component spaces is characterized by the quantitative dimensionality M / N. However different, all modes of construction will result in the same rational dimensionality. This closely resembles the original geometrical picture: it does not matter whether we divide a line segment of the length M into N parts, or prefer to put together M unit lengths $1 / N$. In either approach, representing a point of the original space with a k-dimensional space (where k may well be rational), we get a space of the dimensionality $kM / kN \sim M / N$.

Naturally, formal constructions will only be intelligible in the context of some activity permitting that particular hierarchy. Traditionally, mathematics does not care for the origin of its objects and the inter-level relations; the dimensionality of a space is deemed to be an inherent feature independent of any other spaces. This works fine as long as we deal with the systems that do not change their behavior when included in the context of a different system. On the contrary, one would have to account for contextual dependence if the inclusion modified the constraints, introducing new interactions. Rational numbers can be reduced to integers while we can unfold new dimensions with appropriate constraints. For real-valued dimensionalities, commensurability would no longer hold, and we'll speak about hierarchical embedding, with the processes of one dimensionality developing in a space with a different number of dimensions.

The numerous "theories of everything" will readily come to mind, with their multiple "folded" dimensions. The resemblance is unlikely to be just accidental. However, nature would hardly ever prefer one dimensionality to another, and all kinds of hierarchical structures might be physically possible. To discover them, we'll need to rearrange our activities, to involve non-integer dimensionalities. As a hint, recall the above two modes of the construction of rational spaces: either inner or outer approach. We cannot immediately see certain effects due to the spatial (and temporal) organization of our bodies; still, there is a trick of making such features an inner property of observable systems. Quantum physics provides many well-known examples of such intertwined hierarchies. Here, it should be quite clear that interference of virtual processes is possible regardless of any quantization, in an entirely classical system, as well non-integer dimensionalities enter the game. This might be considered as an indication of the relative nature of the difference between quantum and classical physics.

1998

Dreams and Things

There is mathematics and mathematics. Most scientists work on some quite practical projects, adjusting the formal instrumentation for better performance in the applications (including mathematics). Within this activity, it seems reasonable to explore a number of possibilities in search of the most viable schemes. Yet another group of problems comes from mathematical entertainment, bringing forth all kinds of funny brain twisters of no visible utility. Amateurs can thus assimilate the historically accumulated heaps of knowledge; for professionals, such musings may do as both a valuable pause in the academic routine and a chance to widen the horizon, getting acquainted with the current trends in the adjacent (or maybe faraway) areas of research. Finally, there is fundamental science, seeking for the foundations of everything, to justify the available formalistic technologies, as well as to soothe the doubts of the influential sponsors.

For an outsider, fundamental mathematics may seem a perfect instance of delirium. Deep in the heart, we can understand and concede the passion of a respectable person for making paper soldiers somewhere in the backyard; however, his ambition to thus gather an invincible army and conquer the whole world would certainly drive us to inquiring into the kind of biochemical and neurological disorder he might have. Of course, many cases do not require any intensive medication: with a harmful idiot, mere regular surveillance and minimal behavioral corrections would be enough. Unlike a schizophrenic, who is aware of his disease but cannot control the phantoms of the mind, which makes him suffer, a great mathematician can live in his imaginary world and be entirely happy believing that this is the only possible reality. This dreamer perceives the world around as an imperfect “model” of the stainless truth and accepts only the agreeable side of things.

Occasionally, almost everybody may experience similar dim moments. Still, if I get regretfully crazy to start bombing the academic authorities with, say, some well-developed theories of the budding habits of green three-horned demons, I am sure, in the best, to be ostracized and blacklisted for all times; with a lesser luck, there is a thick prospect of an asylum. On the contrary, some (originally wealthy) guy may significantly improve his well-being studying the intricate aspects of undecidable nontriviality…

A typical fundamental theory is born like that: we have some idea of how a certain class of things is designed; now, let us imaging something that is not exactly the same way and explore the observable deviations. Up to this moment, no serious objection. We do not know what this stranger is like, but we are free to call it names (the tongue won’t break) and play with our paper soldiers to the edge of doom. Persistent complications will spoil the party when the (possibly collective) author forgets about the fantastic nature of such imaginary creatures, and pictures the real world as it has never been. Later authors begin with the already formatted fancy and contrive more freaks, to obtain yet another (presumably more fundamental) abstract theory. Finally, we get what we fought for: a fundamental math entirely alienated from real life (including science).

On the funny side, even such an off-head mathematics will bring about many valuable and useful inventions. The world as it is rules it all, so that any ultimate perversity is only possible within some objective setup. Crazy science is a mirror of a crazy social organization of the present humanity yet lacking the truly human reason. The exaggerated adherence to the rules of the game has become a value in itself millennia ago, when the prehistorical savagery gradually grew into a savage civilization; indeed, in is only in the formal communication that those who cannot yet communicate the human way could be brought and held together. With the fall down of the class economy (provided we manage to witness it some day), everything will get much simpler, at least, because there would be no need to prove anything to anybody.

Note that any (however abstruse) theory grows, in its deepest depth, from a quite earthly root. That is why the admirable inventors rarely present anything dramatically new; for the major part, they just transfer the already known onto their fantastic world, leaving us with the same ideas of a point, a number, ordering, addition and multiplication… That is, everything is decent and proper: the truncated two-valued logic, every species under an appropriate genus, the deductive method, structured artificiality, and an impressive reference apparatus all through. It shines with angelic beauty, but nobody knows what and why it is. As for convincing power, let us keep silent. Today, mathematical proofs do not convince even great mathematicians. The very possibility to arbitrarily distort any (however rigorous) theory is a clear indication of an inner (or even undisguised) slackness. No doubt, there are enough fancy words to explain even that. For example, in nonstandard analysis, they turn themselves inside out, to prove the existence of nonstandard individuals. Just that many queer ultrafilters! Still, after all, here comes a reluctant admission that the ends won't meet without the banal choice axiom (or one of its equivalents). Which is already a kind of standard synonym, a symbol of arbitrariness. While the ill-starred Russel classes have eventually been deprecated among the honest folk, entirely dismissing the axiom of choice would bring the whole enterprise to the final crash.

Nonstandard analysis is to discuss its artificial entities on the basis of a number of *ad hoc* rules. All the construction is intentionally cut to mimic the well-known high-school results, and hence its value is reduced to occasional heuristic hints. Why. Just because there is no real-life trace of any of the exotic objects inhabiting the theory. Our everyday existence is quite rational. Even real numbers are already beyond practical accessibility, and we can only judge about them by indirect evidence, interpolating a chain of observables. Quantum theory is replete with higher-scale infinities; still, they are judiciously said to be unobservable by their very essence, to eliminate the prospects of eventual migrating to some Hilbert space… Does that mean that all the mathematical infinities are primordial evil? Not at all. The logical implication only calls for a little more prudence in making them go, discriminating a sheer fantasy from the established fact.

For an illustration, take the ordinary mathematics of ordinals. Everybody is well acquainted with integer numbers. Used to indicate the order of enumeration, they are referred to as finite ordinals. For every integer, there is an integer immediately following it, and so on, until one gets bored. That is, integer numbers form a uniform scale to measure all kinds of finite things: with any step length, we can get from a the point A to the point B after an integer number of steps. If the last step brings us too far, just use twice as short steps, and so on. Within the desired precision, we are bound to get right where we need. With the appropriate units, the (practical) distance between A and B is always expressible as an integer. In honor of an Ancient Greek, such scales are called Archimedean.

At this stage, mathematicians employ the sleight of hand. Their pet trick is inductive (or recursive) definitions. We are to believe that the existence of the integer number 1 together with the possibility to fancy an increment for any given integer n imply the existence of a recursively defined set N containing all the positive integer (natural) numbers. Every layman will immediately see a striking resemblance of this logic to the axiom of choice: take one element from each set of a given family, and this will produce yet another set. In view of the common treatment of finite ordinals as sets (in the von Neumann scheme) the similitude becomes almost perfect. However, mathematicians will never agree: this is different…

Alright, let it be. With all that, why should feasibility always mean existence? Theoretically, any large enough ground can be used to construct a house. However, if I wish a tiny sweet home of mine on the Concord square in Paris, do I have a slightest chance? So, I'll have to settle elsewhere. If we are sure to imagine an integer number greater than any currently available (and not merely denoted by the character n), this does not make such an integer a sure thing. Quite possibly, this new entity has not yet been involved in any practical activity, and hence it remains merely plausible, rather than actually constructed. Some computer, for instance, might object that big numbers cannot be represented within the present architecture, and hence they cannot be really used and should be qualified as only potentially existent. One must be a mathematician, to mix up imaginary and real things.

Once again, let as swallow the far-from-nice tricks and admit the existence of a communion of all the integer numbers. But why should it necessarily be a set? After the experience of Russel paradoxes, it seems to be commonly accepted that proper sets are all contained in a standard universe. All the rest should be referred to as classes, with a significantly less rigorous treatment. From nonstandard analysis, one learns that the collection *N* cannot be an "internal" set (that is, it does not belong to the standard universe). Still, following the old tradition, it retains the status of a set (albeit "external"). Similarly, real numbers are postulated to form a set, and the set of the subsets of a given set is a common construct… Such verbal exercises are not innocent. The perverted game of infinites is rooted right there.

Now, accepting *N* as a set, it is logical to ask: how many elements does it contain? Obviously, this number is greater than any given integer, and a different scale is to be used. We are told that the number of all integers is infinite, but we are free to invent some handy name and notation: let this particular infinity be denoted as $\aleph_0$ (assuming that there can be other infinities, to be labelled accordingly). One could wonder why we need yet another sign for infinity, in addition to the centuries-old character ∞. The answer is: to make the game funnier. Anyway, it did not have to do anything with reality, from the very beginning. In other words, there is infinity in general (∞) and the specific types of infinity, cardinal and ordinal numbers.

Since, by definition, the "set" *N* is greater than any finite number, it will represent an ordinal number greater than any finite ordinal. In this context, it is commonly denoted as ω. Meaning that, once mentally introduced, it is certain to really exist. More freaks, more fun. By analogy with the natural numbers, one might conjecture the existence of the next (infinite) ordinal $\omega + 1$ which is greater than ω. Then, once again, the induction trick is to ensure the existence of the set of the elements of the $\omega + n$ form, for all (finite) integer n. As an expectable continuation, there is the ordinal $\omega + \omega = \omega \cdot 2$, which can be incremented in its turn… This immediately supplies all the combinations like $\omega \cdot k + n$, and then those like $\omega \cdot \omega = \omega^2$, and all the rest, following the same scheme. A real killer! Just keep playing on, up to the new countable type ε_0, and even greater curiosities, finally coming to the first uncountable infinity ω_1 … As soon as an axiomatic layout is introduced, the dreamland becomes exact science, a matter of professional pride and a high-profit business.

For a passer-by, everything looks utterly strange. Well, with minimal reserve, one can accept the axiom $\alpha + 0 = \alpha$ for all ordinals; on the contrary, the axioms like $\alpha \cdot 0 = 0$ or $\alpha^0 = 1$ raise an inner protest. Since high school, we have a custom to treat $\infty \cdot 0$ and ∞^0 as "indeterminate forms" that should be resolved using some additional information specific for a particular situation. Further, ordinal addition is said to be non-commutative:

$$\omega + 1 > \omega,\ 1 + \omega = \omega$$

In the general case the sum is to be defined in a crazy manner as

$$\alpha + \gamma = \sup\{\alpha + \beta \mid \beta < \gamma\}$$

It is generally known that the existence of the exact upper limit in far from being a trivial circumstance in the world of infinities; roughly speaking, one cannot properly define it without the axiom of choice (at least implicitly invoked). Of course, when our paper soldiers are to fight exclusively in the conditions of the *a priori* applicability of all the assumptions, there are no objections. That will do fine for a game. In science, such a generous environment can hardly ever happen. We don't need a theory of nobody knows what; we are to provide high-performance tools for everyday usage. The applicability of the formalism is then related to the object area rather than any good intentions.

As nonstandard analysis indicates, the numbers of the $\omega - n$ form are also infinite, as compared to finite ordinals. The indeterminate form $\omega - \omega$ is understood as yet another hyperinteger number which is less than ω, but still greater than any finite integer. The class of hyperinteger numbers is, therefore, dense everywhere: between any two infinities, there is yet another one. This is more compliant with the standard of a complete ordered filed; on the other hand, the existence of all those infinities remains a matter of subjective conviction.

A lay person will logically wonder, why, for infinite ordinals, we are to sacrifice the commutative feature of addition and multiplication instead of some other properties of the ordinary numbers. Admittedly, the expectable row of infinities will no longer be a field in the classical sense, and we need to weaken our axioms. But why not define addition in some other manner? For instance,

$$\alpha + \beta = \sup\{\alpha' + \beta' \mid \alpha' < \alpha \;\&\; \beta' < \beta\}$$

Thus defined addition is obviously commutative and has really wonderful properties: $\omega + 1 = \omega$, $1 + \omega = \omega$, $\omega + \omega = \omega \cdot 2 = 2 \cdot \omega = \omega$. In this view, the whole zoo of countable ordinals would be a sheer fiction: there is a single countable infinity, and no need for extra junk in the head. Then, similarly, we can introduce the negative infinity $-\omega$, so that $-\omega - n = -\omega$. In the result, there is a well-ordered field with the only little exception that the inequalities $n + 1 \geq n$ and $n - 1 \leq n$ are not always strict. In any case, this is a much weaker assumption than broken commutativity! Theoretically, one could demand that $\omega - \omega = 0$, $\omega + (-\omega) = 0$. Then our system will be closed in respect to addition and subtraction. However, the correct decision would rather follow the usual rule of calculus and consider such expressions as indeterminate terms of the $\infty - \infty$ type, which are to be resolved on the basis of practical considerations.

So, the fantastic worlds invented by reverend mathematicians are of no use for an ordinary person. There are the positive and negative infinities, and the infinite limit can be defined in a quite natural way. Similarly, there is the only infinitesimal value $1 / \omega$ (the upper limit), complemented with the single negative infinitesimal $1 / -\omega$ (the lower limit). We always keep within the standard analysis and contemplate the harmony and beauty of the only world.

There is yet the specific issue of the existence of the uncountable infinity $\aleph_1$ (or ω_1), which should be reconsidered elsewhere. Still, it is clearly predictably that things are much simpler than the academic compendiums narrate. To start with, our "symmetrized" definition of addition is perfectly applicable to real numbers too, which means that the universal countable ordinal ω neatly coincides with the uncountable infinity ω_1 and all the other uncountable ordinals! That is, uncountable sets qualitatively differ from countable, rather than on some quantitative grounds; basically, this is the opposition of discreteness and continuity, which can gracefully coexist in a bounded area, like, say, the interval (0, 1). As a bonus, one comes to the comprehension of the different levels of countability: the uniformly dense set of rational numbers is obviously an entity of a different kind than the sparsely spaced natural sequence. In these lines, many great ideas are yet to come.

Just learn somewhat better manners while working with sets and their mappings, and do not much trust the words "and so on" (in the inductive sense); this will tame higher-order infinites and bring them in good terms with reality. On such a foundation, no quanta can entangle us, and the light barrier is no obstacle.

1997

Crazy Kantor

Since the earliest school experiences, we know that numbers can be either natural or real. All the rest (rational, complex numbers, quaternions, octonions, transfinite numbers, ordinals *etc*.) can only be called numbers in some less obvious sense, as they basically appear as specific structures upon the true numerical foundation (probably with some recourse to topology). Even real numbers are indeed a source of doubt and confusion: the speculations about the limit transitions in the bundles of rational numbers can hardly convince anybody with a basic mathematical background, as the thought of a final result of an infinite process seems to be an apparent logical fallacy, a kind of wrong qualifier. As it is well known, real math came to support the everyday activities (like land measuring, comparing volumes and masses). The relation between real and natural numbers is therefore quite similar to that of the measurable and the measure, an existing thing and its perception. And these opposites are certainly distinct. We are well aware of the fact that our notions of the world are necessarily limited, restricted to its minuscule portion that needs a practical effort here and now. There are hidden motives, and there are explicit goals. As well as an ocean of whatever is beyond the scope of a particular activity. Hence the idea of discreteness and continuity, which are irreducible to each other just because they refer to different object areas that cannot be compared in a straightforward manner.

On the contrary, the academic mathematics accepts from the very beginning that there are no constructs differing from each other to any significant extent, so that their interrelations are purely formal, and one thing can always be reduced to (or deduced from) another. The unmeasurable richness of nature is thus squeezed into something flat and barren. The mathematical picture of the world is extremely rough, approximate, incomplete. Such immodest abstractions still can be useful sometimes. However, the generality of reduction and its imperative ubiquity are sheer illusion. Logical faults and the vagueness of the fundamental principles are the price to pay for unwarranted ambitions, blind enthusiasm and pretentious exaggerations. As the other side of the same, too much scruples about the form and the standards; in the civilized society, the ostentatious courtesy and dress code serve as a common disguise for mutual hatred and instinctive vulgarity.

As soon as we drive out the distinctions between integers and real numbers, there is the question of the very their existence: what are they? do they refer to assumingly different constructs, or are they merely two representations of the same? The education standard insists on the traditional answer: there is absolutely no way to map natural numbers to reals, and the other way round. This is what the great Kantor has demonstrated with all the possible rigor. Which discovery has brought him in an asylum for the rest of his life. To be sure, the equality of infinities is to be commonly understood as isomorphism, the revertible mapping of one onto another preserving the essential properties of the both.

This latter formulation is already a prompt target for skepticism. In Kantor's epoch, even the greatest mathematicians would not unanimously vote for the actual existence of any infinities. What is a mapping of one infinity onto another is no more obvious. Nothing to say about the criteria of importance of any properties. To overcome the lack of harmony, mathematical philosophers have invented a peculiar mode of speech allowing to discuss the incomprehensible in an entirely formal tone, regardless of the sensibility of such a science. This obligatory platform proudly bears the name of axiomatic approach, and every person of any age and race must conform to the standard on penalty of permanent excommunication from mathematics, and science in general. Still, sweeping the litter under the carpet (or the avoidance of the thought of a white monkey) can never withdraw any real problems. And logical problems in particular. Those that make the allegedly rigorous mathematics utterly non-convincing.

Let us recollect the textbook-haunting diagonal scheme of that Kantor's proof. First of all, real numbers (conventionally restricted to the segment [0, 1]) are to be identified with the sequences of binary digits, 0 and 1. We are told that every sequence of that kind represents a real number in the binary positional notation, and that every real number is therefore representable in this form. Well let us skip the questions about the admissibility of this two-way logic and leave aside the very nontrivial nature of the positional notation (or a "fundamental sequence", as Kantor put it), as well as the issue of the identification of a thing with its name. Just note that the very possibility of matching one infinite sequence against another is inherent in the very mode of problem formulation. Now, assume that all the real numbers can be enumerated, that is, put in correspondence to the elements of the naturally ordered naturals. Once again: as such an enumeration is infinite, there is an issue of the feasibility of establishing the correspondence between a binary sequence and its number; so far, we voluntarily suppress the discussion. It is also assumed that, for the sequence labeled with any natural number n, it is possible to determine the value of the n-th digit (which will admittedly be either 0 or 1). One can easily revert this value, taking 0 instead of 1, and 1 instead of 0. That is, there is an exact correspondence between the number n and a binary digit, which gives a binary sequence that cannot coincide with any of the earlier enumerated sequences, since (by construction) it differs from the n-th sequence in the n-th position. According to the original assumption, any binary sequence (including the "antidiagonal" sequence just constructed) corresponds to a real number; however, none of the earlier assigned sequential numbers would correspond to our antidiagonal, which (obviously?) contradicts the assumption of the complete enumeration. This is to provide the substantial grounds for rejecting the hypothesis of the enumerability of real numbers, thus extending mathematics to however big "cardinal numbers" as the levels of actual infinity.

One could quite reasonably ask why, with several highly problematic assumptions for the premises of a deductive chain, the resulting contradiction should be interpreted to refute only one of them. Whence the immunity of the others? Our respect for an old tradition, or even earlier derivation of some statements as theorems in another theory, would in no way influence their vulnerability in another context, possibly inappropriate. With all the adherence to the axiomatic method, one can cancel any mathematical truth at all: a shadow of doubt about one of the axioms or the validity of the derivation scheme is quite enough. As it often happens, a seemingly innocent rearrangement, an extra brick, may bring down the whole construction. In the same way, a single contradiction is enough to ruin the whole of a mathematical enterprise rather than just a last minute extension.

For instance, take the convention that picking a certain digit from each of an infinite number of infinite binary sequences will produce an infinite sequence of the same kind. This is a direct reference to the well-(or ill-)known axiom of choice, whose admissibility still remains a cause of ardent controversy. To illustrate the nontrivial nature of the problem, let us simply reformulate Kantor's proof for transfinite numbers. Indeed, every natural number, just like real numbers, can be represented by a sequence of binary digits, with the only difference that, for integers, this sequence is "finite", that is, all its members are equal to zero after some sequence number (note that there is no effective procedure for establishing the fact of this finality in each particular case). Now, let as assume that all such sequences can be enumerated. Reproducing the same diagonal reasoning, we obtain a "number" that does not belong to the natural set. What's the difference? Compared to the original Kantor's deduction, here, we have originally enumerated only "finite" sequences, while the antidiagonal sequence is "infinite". It could still be considered as a representation of some integer number, which happens to be "infinitely big", and the existence of such numbers leads to a well-developed transfinite mathematics. Getting back to the school version, one is prompted to admit that, applied to some "countable" real numbers, Kantor's diagonal construction would produce a sequence representing a real number of a different kind, an "uncountable" real. This also gives way to a meaningful mathematical theory distinguishing the levels of "reality" by their inner complexity. Why not? We speak of transcendent numbers as different from mere irrational. More intricate distinctions might be useful in certain applications.

Of course, there are as legal conceptual alternatives. Let's fancy that the antidiagonal is similar to all the other binary sequences, and hence it must correspond to a traditional real number. With all that, this sequence is not entirely alien to any enumeration at all. Indeed, it can easily be included in the original enumeration: just increase all the original numbers by 1 and assign (ordinal) number 1 to the antidiagonal. Repeating the diagonal trick once again, we'll obtain yet another "extra" number, which still can be enumerated in the same manner. That is, real numbers cannot be enumerated once and forever, but there is no real number that would not be present in one of the possible dynamic enumerations. Real numbers are therefore both enumerable ant not enumerable. This contradiction indicates that, for infinite sequences, the very notion of enumeration needs better elaboration.[11]

Just think about how the elements of a finite set could be enumerated. One would drive a hand into the sack, rummage an element out, and look for an (ordinal) number to attach. The procedure allows (at least) two modes of continuation. Thus, the exhaustion technique demand that we put the captive elements in a different bag (another set) and proceed with extracting the elements from the original set until it gets empty. In this approach an enumeration is represented by a sequence of non-intersecting pairs of sets, which is a very complex structure (often referred to as a partition of unity). An opposite

[11] One will readily observe the kinship of our renumbering method to the usual renormalization techniques of quantum field theories. Such renormalization is nothing but the extraction of the self-action (an operator diagonal) of a physical system to a separate branch that needs a different treatment to bring its contributions to the scale comparable with that used in the rest of the series.

choice is to feed the just enumerated element back to the original set and pick the next candidate from the same company. If the catch is already numbered, leave it as it is; otherwise, attach the next label and go on. The enumeration is then formally represented by a stochastic process, a probability distribution. In whatever case, there are certain objective complications. Sorting out the remainder is a poorly defined and slow-convergent procedure. In the first method, there is a problem of hunting the elements in extremely rarified media; the stochastic approach can never tell for sure if all the elements have been enumerated, or some statistically elusive individuals can still be present. When the elements of a set are from the material domain, they may sometimes be spoiled by the impact of enumeration. For example, if, in the exhaustion scheme, a catch of living fish is further kept on the ice (for the destination set), live numbers get gradually transformed into dead numbers; though the second method seems to be a little bit more humane, the numbered fish life time is not infinite in any way, and it may pass away before we are ready for the definitive verdict.

Well, the degree of animation of the object area may be of no importance in a particular theory. Still, this bring us back to the issue of essential qualities, which does not promise to make our problem any more tractable, since the theory is thus augmented with the necessity of formal derivation of the criteria of applicability.

With all that, we are inclined to think that the generalization of enumeration to infinite sets is a very complicated task hardly ever having a clear and unambiguous solution. Admit that somebody has invented a technology to enumerate the elements of a set in a quite definite manner, without recourse to random choice. That is, there is an effective procedure for computing the sequential number for all the numbers from a given class. Who can warrant that there no other numbers, of a different kind? Provided a new class has been discovered (or constructed), we need an appropriate enumeration scheme, as well as a rule for merging enumerations. Isn't is much simpler to admit that no infinity refers to any actual existence, meaning a currently developing process (possibly without end)? In this sense, the finite means the already completed. This is our present, while the past and the future belong to infinity.

To be honest, the situation is not that simple. The hierarchy of the present contains the appropriate levels to represent the past and the future. Similarly, in any infinity, there are finite structures, a kind of projection of the present, a shift of the reference point (compare the grammatical tenses like *future in the past* or *future perfect*). However, this matter is to be discussed elsewhere.

Get back to Kantor. We cannot tell beyond doubt, whether the lack of comprehension among the colleagues drove the poor guy to madness, or his innate insanity has ultimately infected the future scientists with crazy abstractions. Probably, the both ways. Still, it is absolutely clear: building a set theory for all in all, means building nothing at all. Unless we have agreed upon the nature of the entities that can become the elements of our sets, so that formal constructs would always imply an objective meaning, there is nothing to talk about, and no science but sheer meaninglessness. There is no reason for preferring one way to another, and no way to tell the right from the wrong. The objective perspective does not exclude the sets of abstract ideas; however, these ideas are to be naturally compatible, bundled by the inner logic of the object. To be sure, every object, beyond all the rest, exhibits the unity of opposite aspects, such as finality and infinity, discreteness and continuity, limits and unboundedness… Adapted to the mathematics of numbers, such universal, primary oppositions could logically be introduced in the theory from the very beginning, say, in the form of a natural sequence and a real axis. There is nothing to prove; this is the starting point for any further development. Armed with such a basic vision, one will get engaged in an honest investigation of the practically useful corollaries, or seek for a freaky bit, to come to a different object area, yet another aspect of reality. The attempts to formally substantiate what is determined by the objectively given order of things are nothing but a kind of ill soul-searching, warped reflection, a logical circularity and deductive self-reference, primitive diagonal reasoning. That is, a perversion and mental disease. Nevertheless, it's still for good, if crazy Kantor will lead the humanity to acquiring more respect for sound practical considerations.

Nov 1984

A Broken Parity

Somebody has once said that a philosopher is a person capable of discovering an inacceptable complexity in the simple and generally accepted. One could consider the following mathematical illustration, on the secondary school level.

There is a notion of the partitioning of a set in a number of non-intersecting classes. That is, each element of the set is believed to belong to some class, but it cannot belong to several classes. One could fancy many such decompositions, which, in general, have nothing to with each other.

There is yet another notion: a symmetric, transitive and reflexive relation could be defined on the same set, which is traditionally called *equivalence*. The equivalent elements do not necessarily coincide; they only are treated as interchangeable in this particular relation. Of course there are as many kinds of equivalence as you can invent, and one may be very unlike the other.

Straight away, a school teacher proves that every partitioning of a set determines some equivalence, and conversely, the classes of mutually equivalent elements of a set necessarily provide its specific partitioning. In this picture, partitions and equivalence can be observed as merely the two modes of speaking about the same.

At that peaceful moment, a philosopher breaks in and immediately starts to wonder whether any set at all can be represented as a number of non-intersecting classes, and whether any set allows for a globally defined equivalence. For, if the notions are not universally definable, there may be doubts as to the admissibility of their comparison, nothing to say about any formal proofs.

The questions are phenomenally stupid. If we have deliberately composed the original set as we like, who can prevent us from picking out a few arbitrary elements to produce a subset? Then, everything that does not belong to this subset will form its complement, so that the subset taken together with its complement would provide a constructive example of a formally correct partitioning. Just the same holds for equivalence: take a few elements, call then equivalent by definition, and let the rest be treated as equivalent among themselves, but not equivalent to any of the chosen few. That is, at least two classes of equivalence can be constructed in any case; more sophisticated constructs are easily introduced using the three magic worlds: *and so on*… Or the three magic letters: *etc*. Or just three points in the end of a phrase.

The sour look of our philosopher indicates that he cannot get the point and remains utterly perplexed. “Good grace!” he pleads, “just think that non-belonging to one set does not mean belonging to any other. If this particular object is not a nail, it does not need to be a screw! There are lots of other kinds of ironware, like pins, dags, bolts, clinchers, or anything at all that we may unaware of until it eventually happens to come across. Claiming that there is nothing in the original set that would not belong to either the chosen subset or its complement, you implicitly assume that the set is *already* split into two non-intersecting parts, in which case the sought-for partitioning certainly does exist! This, however, does not exclude any less trivial occurrences.”

In the absolutely the same manner, the possibility of explicitly demanding the equivalence of some elements of the base set in no way implies the mutual equivalence of the rest. The usual “proof” takes for granted that the non-equivalence of two elements to a given element different from the both means their equivalence. Isn’t it a somewhat too strong assumption? Its point is to establish an interdependence between two quite different relations, which may only hold on rather restrictive conditions that need to be properly explicated; in fact, once again, you implicitly postulate what you are going to deduce.

Consider a very simple example. Let us admit that, in our theory, only the open balls in some space can be taken for sets; obviously, none of these sets can be split into any number of non-intersecting sets (in the same topology); moreover, even a finite (or countable) cover may not always be possible. To introduce partitioning, one needs to extends the original universe, which may lead to a quite different mathematics. Similarly, any equivalence taken as it is (as a set-theoretic relation) does not refer to any other relations (including any non-equivalence). We can always decide whether these two particular elements of the set are equivalent or not, and all the rules of equivalence will perfectly hold, so that each of the mutually equivalent elements could represent the same class of equivalence; but, in general, we do not know if there is a collection of representatives representative enough to allow the corresponding classes cover the whole original universe.

This is what elementary logic has to say. In practice, we can always restrict ourselves to the “right” sets behaving as expected, to ensure the validity of deduction. Still, such “proofs” can hardly be any convincing, as all we can is to formally construct the objects of a given class, stretching the theory to the desirable result.

Is there anything to blame? As a matter of fact, all the sciences do exactly the same, trying to adapt the inner structure of the science to the natural phenomena. Demystified (objective) mathematics is to take its place among the other sciences, instead of remaining a sort of religious revelation; this will make it much more friendly and ready to accept all the signs of our profound esteem.

Well, certain consequences may seem to be less comfortable to a working mathematician. For instance, the so much worshipped proofs by contradiction will no longer be treated as rigorous proofs, but rather as convenient heuristics, the public motives of the important decisions, whose truth is yet to be sought for elsewhere. Indeed, even provided we have found what this particular thing is not, we still have to guess what it really is. Negative definitions constitute an important stage of any cognition: they are a kind of search activity, overall orientation. However, since all the other modes of deduction become entirely conventional and do not lead to anything but working hypotheses, there is nothing tragic in this inherent insufficiency at all: we do what we do, produce what we can, while it is up to the practical implementations to put things right and eternalize the truly valuable.

A mathematical theory grows like any other: the empirical considerations outline a certain class of phenomena to cope with (the object area, the “universe”); we decide which features of the objects should be of primary importance in this context, while all the rest is to be somehow (in an as formal manner as possible) related to the that basic foundation. That is, there is no goal of “proving” anything, or to be convincing enough; a theoreticians is just to *relate* one thing to another, bring things together, rather than deduce. When the object area is structured enough, it may be quite admissible to reason by analogy, turning the already found regularities into formal schemes allowing both to predict something on the basis of the available facts (thus putting forth a hypothesis) and, conversely, to suggest a range of formal solutions for a given practical problem (the activity of justification). This is much like a physicist plans a series of experiments or, say, concludes about the inner structure of a quantum system by the observable spectrum. Well, the same can happen to everybody in everyday life: finding a crack in the wall, we envision the possibility of further destruction and try to collect the tools and materials necessary to patch it up. It is self-understood that some happenings won’t lie in the line of our theory, and we’ll need to consider a different methodology.

The dry residue: the traditional mathematical approach is quite useful in the circumstances similar to those that lead to the present state of affairs; the object area is assumed to be broad enough, as we position ourselves far from the its natural boundaries. However, since no scheme can be absolutely universal, a grain of humor may serve for good in assessing the results: normally, we obtain what we intend to obtain, while nothing prevents us from getting anything else when things change.

Now, let’s get back to partitions and equivalence. The school attitude employs the principal postulate of the modern mathematics: what can be built has already been built. Mathematics does not distinguish actual existence from mere possibility. In particular, any set is prepared for us beforehand, just patiently waiting for a glimpse of our attention and a condescending couple of words. This means that no set operations are capable of producing new sets: all they do is to relate one existing set to another. Thus, speaking of a union, or an intersection, we only express the fact that two sets (from some static universe) are related to another set; similarly, the idea od a subset is to convey a special mode of linking one set to another. All such bindings have long since been established, and we treat them in a static manner, as ready-made. Formally, such structures are introduced using the intuitive (undefinable) construct of an “ordered pair”, so that any relations between sets could be represented by the sets of ordered pairs.

As one can easily observe, the union (or intersection) of two sets is not always conceivable, since this possibility depends on the presence of a set of the same universe that could be associated with the union (or intersection). For instance, in the above universe of open balls, the union of two balls is only definable for one ball embedded into another; the union of two non-intersecting (or partially intersecting) balls will no longer be a ball. The same holds for the intersection. In this case, for any embedding, the union and the intersection are to select the outer and inner balls respectively: they become the operation of *projecting* an ordered pair onto one of the components (similar to arithmetic maximum and minimum).

Consequently, the theoretically deducible properties of sets are entirely dependent on the structure of the universe, so that all the sophisticated theorems merely reconfirm what we have already assumed by the very choice of the object area. In particular, the existence of a partitioning is out of question if its components do not belong to the original universe.

On the other hand, since any relations are formally defined as sets, the proof of the close correspondence between partitioning and equivalence is sheer tautology: the validity of a static notion of equivalence is based on the assumption that the corresponding classes have already been built.

Yet another traditional approach starts with the algebraic shift of the focus from the objects to the possible modes of operation. The background universe (a configuration space) is still to be involved; however, in general, it does not need to be a set, and all we demand is the existence of elements combinable through a number of (informal) manipulations. We do not care of what an element, or an operation, exactly is; it is quite enough to consider the overall features of this manipulative activity as the only available deductive scheme. Just for convenience, it is often assumed that the operations are defined for all elements of the universe; what does no suit us is to be deliberately put aside. In the presence of several operations, their domains may differ; still, it is always possible to restrict theory to the commonly definable, possible allowing for a few singular points. Such theories are known as complete. Obviously, there is nothing to expect in the outcome beyond sheer tautologies. Why not? One is free to play with void forms for one's personal entertainment.

Algebraic structures tend to smoothly glide towards the idea of representing operations with elements, and elements with operations; the usual set-theoretic methods then work there in full. One of fundamental tricks in this approach is to postulate the actual existence of any domain. For instance, each element invokes all the elements involved in its production via a (presumably) fixed combination of operations (a function). In the assumption that all the possible functions have already been evaluated, such prototypes are readily joined in a static object, a class of equivalence. The built-in completeness immediately recalls the existence of covers or partitions.

Some algebraic structures introduce a kind of partitioning from the very beginning. For instance, the bulk of the abstract (informal) elements can be complemented with an already known algebraic structure (or a set). This leads to discriminating functions by the type of the object returned: basic operations always return an element of the "syncretic" part of the universe, while some functions calculate to a "number" (an element of the structured field). This does not change anything in principle. The admissibility of treating a domain of a function as a class is a very strong assumption that virtually "preinstalls" all the deducible properties in the theory.

An object-bound mathematics would consider a definite (though not always formally definable) kinds of abstract objects, aiming at an explicit description of the possible regularities (links) as a sort of interpolation schemes to conclude on the properties of anything that is constructible in compliance with the rules of the game. All the answers are implicitly presupposed in the static case. There is no significant difference between the statements of all types, and no reason for obstinately preferring some schemes to the others. The logic of theory is to comprise all the diversity of formal tricks, and any level of deduction is to be built upon all the rest, since it is only in their integrity that the object of the science is expressed. Thus, the usual habit of operating with "logical" junctions (**not**, **and**, **or**…) in no way means that this "symbolic" logic could exist on itself, producing some "pure" knowledge regardless of the object area. In each case, we only try to control our objective acts bringing them under the familiar forms; this initiates an activity of reinterpreting the abstract logic in terms of object manipulations, which drives us to rearranging the object area to comply with our behavioral stereotypes. Some objects would not fit in the template; in this case, we modify the scheme, effectively replacing one logic with another. Fortunately, the continuity of scientific development rarely comes to any radical revision, and many techniques get productively employed in the course of several millennia (albeit with innumerable refinements and specifications). Still, search for new paradigms may be of crucial importance sometime.

Our cognition develops from syncretism, an overall impression and intuitive judgment, towards all kinds of distinctions, and then to the attempts of recovering the former entirety, reconstructing the whole from the diversity of the disparate components. The modern culture, with its market economy based on the universal division of labor, won't allow to complete the synthesis, and science mainly turns around grouping and grading, with the class inequality as a generic theme. Hence all those decompositions and partitioning in mathematics; we need equivalence just to stress the differences, and all the significant features are normally introduced in respect to the quotient sets, that is, for classes rather than elements. What is incompatible with any ranks is out of science (and the civilized society as well). The world of the future, presumably inhabited by perfect singularities, will require a different mathematics. Any alliance is to become transient, relative, limited to the demands of a specific problem, while extensive task swapping will cause frequent rearrangements in the object area. This is one of the determinative principles of the hierarchical approach.

As earlier indicated, the object-bound attitude is implicitly laid in the foundation of any formal universe, which is absolutely necessary to reconcile the theory with the practical needs. The next step to take is to abandon any "final" definitions at all, admitting a "natural" diversity never subject to any universal formalization. At any rate, the very idea of formalization implies a partial treatment of many special cases. A true science is not to invent any theories of everything; it must be capable of producing a straightforward and minimally complicated theory for each particular aspect of the object area, with a commendable soberness of mind preventing us from exaggerating our ingenuity.

In the example of sets and equity, the bent for complete classifications (partitioning a set into non-intersecting classes) might yield to the discussion of various *scales*, the collections of zones, with the elements of the universe tractable as relatively equivalent within each zone. Why relatively? Because every scale is hierarchical, with the scales of different levels never reducible to each other. Equivalence on one level may well account for qualitative distinction on another. For example, in music, the same note can be pitched a little higher or lower (within the same pitch zone), depending on the musical context and the artistic sense; similarly, in dancing, the same typical figures allow for quite different arrangements. Well, this is what makes the essence of art; still, in science, there are many levels of formalization, and one cannot easily tell a fundamental theory from a semiempirical model.

No scale can embrace everything; still, the uniformity of the universe admits hierarchies unfolded in full starting from any individual element. This uniformity is quite like the inner symmetry of the scale, the equivalence of the elements within a zone. Normally, any "outer" symmetry means the existence of the level of the scale, where this invariance comes up as zonal equivalence. In this way, the scale can be uniformly extended to the whole object area; completeness is thus restored, on the hierarchical basis.

Every time we consider a subset of a given set, a static scale is really meant. It comes to the neighborhood of a given element, its close enough vicinity, while the rest is beyond the limits of measurability. Another subset will determine a different set. To relate one subset to another, we have to introduce yet another level of hierarchy containing the object of a new kind, associated with some inner hierarchy. Thus, in music, harmony can be treated as a hierarchical structure, a union of several zones; on the other hand, polyfunctional notes exhibit an analog of set intersection. Such "local" hierarchies are not generally reducible to the incident elements and scales. It is only in a static context that a set becomes the collection of *all* its elements, while an element can be defined as a collection of *all* sets containing it. A subset and its complement are defined through each other in the exactly the same way. The fundamental logical problem is to somehow restrict the essentially non-formalizable notions of *all* and *everything*; the many variants of specification lead to quite distinct theories.

When it comes to creating and destroying mathematical objects, the issues of consistent formalization go top. For instance, people keep being born or gone, and there is a set of living people at every moment (in some conventional scale); however, there is no static set to contain every person at all, as there are at least those who did not yet come to life. Hence any theories of universal values (including the universality of mathematics) are beneath all criticism. Other object areas reveal similar peculiarities. Thus, electrons may form a Fermi gas; but, under certain conditions, they get stuck in Cooper pairs, forming a Bose condensate, the state of an entirely different symmetry. Our notions of food and dwelling are historically mutable, which makes the sets of culinary ingredients and construction materials as flexible. In the same line, what can prevent us from considering algebraic structures where the admissibility of operations depends on the elementary composition of the base, as well as the elements may change in the course of certain operations? What shall we take for the analogs of partitioning and equivalence in that case?

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